Pricing Spread Options by Generalized Bivariate Edgeworth Expansion

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A spread option is a contingent claim whose underlying is the price difference between two assets. For a call, the holder of the option receives the difference, if positive, between the price difference and the strike price. Otherwise, the holder receives nothing. Spread options trade in large volume in financial, fixed-income, commodity, and energy industries. It is well known that pricing of spread options does not admit closed-form solutions even under a geometric Brownian motion paradigm. When price dynamics experience stochastic volatilities and/or jumps, the valuation process becomes more challenging. Following the seminal work of Jarrow and Judd, we propose the use of Edgeworth expansion to approximate the call price. In the spirit of Pearson, we reduce the cumbersome computation inherent in Edgeworth expansion to single numerical integrations. For an arbitrary bivariate price process, we show that once its product cumulants are available, either by virtue of the structural properties of the underlying processes or by empirical estimation using market data, the approach enables analysts to approximate the call price easily. Specifically, the call prices so estimated capture the correlation, skewness, and kurtosis of the two underlying price processes. As such, the approach is useful for approximate valuations based on Lévy-based models.

Keywords: Spread options; Edgeworth expansion; Lévy processes

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1. Introduction

A spread option is a financial contract between two parties. For the energy/power and commodity markets, spread options are in the bread-and-butter tool kit for risk management. In this paper, we consider a spread option of European type. In this case, at the maturity of the option, the holder of the option has the right to exercise the option and collect the payoff which is difference between the price difference between the two underlying assets and the strike price. If the payoff is negative, the holder forgoes the right. Spread options trade extensively in organized exchanges and between counter parties. Its popularity many times stems from the need to hedge the price difference between the two underlying assets and/or speculative motives between two parties with differing beliefs about future price movements. It is well known that pricing of spread options does not have closed-form solutions even under a geometric Brownian motion (GBM) paradigm. Various approaches on pricing spread options center around two age-old thrusts - bounds and approximations. Earlier work on the subject include those of Shimko (1994), Kirk (1995), Pearson (1995), Mbanefo (1997), among others. In the recent past, Carmona and Durrleman (2003) provided a procedure requiring the solution of nonlinear equations for estimating price bounds. Dempster and Hong (2000) and Hong (2001) employed the Fast Fourier Transform for valuation of spread options. Deng, Johnson, and Sogmonian (2001), Hikspoors and Jaimungal (2007), and Benth, Benth, and Koekebakker (2008) studied the pricing issues in energy markets. Li, Deng, and Zhou (2008) and Bjerkund and Stensland (2008) gave two different closed-form approximations under a GBM paradigm. A very efficient numerical procedure using Fourier space time-stepping approach was suggested by Jackson, Jaimungal, and Surkov (2008/2009). In addition, another Fourier transform based approach for numerical solution of spread option appeared in
this journal was proposed by Hurd and Zhou (2010). Kouam (2007) explored
the use of Monte Carlo simulation to study scenarios where the bivariate price
process contains stochastic volatility and jumps. The extension of the underly-
ing paradigm to one that includes Levy processes has been considered by Benth
and Kettler (2011) in the context of spark spread.

For a European option, the prices at maturity are all what is needed for
pricing. Hence the stochastic process governing the price dynamics is relevant
only to the extent it produces the price distributions of underlying assets at
maturity. For an option with a single underlying asset, under the classical
Black-Scholes-Merton paradigm (1973, 1990), the price at maturity follows a
lognormal distribution. It is widely acknowledged that market data deviate
from such a paradigm and with a consequence of frequent and serious mis-
pricings. To circumvent this difficulty, Jarrow and Rudd (1982) pioneered
the use of Edgeworth expansion to approximate arbitrary price distributions at
maturity using the lognormal as the approximating distribution. As a result,
the skewness and kurtosis of the price process are reflected in the option price
so computed. The term Edgeworth expansion was coined after Edgeworth
(1904) in his study of law of error. Its use for approximating probability dis-
tributions spans more than one hundred years in the statistical literature (e.g.,
see Cramér (1946), Kendall (1949), McCullagh (1984), and Stuart and Ord
(1994). Following the lead of Jarrow and Rudd, others have used the approach
for applications in option pricing - particularly options with a single underlying
asset. These include the work of Duan, Gauthier, and Simonato (1999) under
a GARCH framework and Turnbull and Wakeman (1991) for pricing average
options, among others. In the past fifteen years, the use of Edgeworth expan-
sion in financial mathematics has dwindled to a trickle. Notable exceptions
are the papers by Zhang, Mykland, and Ait-Sahalia (2011), and a thesis done
at Imperial College, London, by Ehrlich (2012) for pricing basket options with smile.

In this paper, we present an approach for approximating the price of a spread option by bivariate Edgeworth expansion. In Section 2, we introduce a few needed notations and results pertaining to product cumulants. In Section 3, we present the bivariate Edgeworth expansion and state its implications in computation. In Section 4, we address the issues concerning pricing spread options. The derivations of the results extend the idea of Pearson (1995) (also, Ravindran 1995). Simply stated, it exploits the law of total probability in computing expectation. We will see that after elaborate manipulations, the potentially cumbersome computation boils down to single numerical integrations. Moreover, numerical differentiation embedded in the expansion is completely avoided. In Section 5, we demonstrate the application of the procedure for correlated price processes that include jumps of the type considered by Merton (1976). In Section 6, we present another example based on the double exponential jump diffusion model of Kou (2002). There, the model enables us to capture the asymmetric heavy tails of returns resulting "volatility smile." For an arbitrary bivariate price process, we emphasize that once its product cumulants are known, either by virtue of the structural properties of the underlying processes (as illustrated in Sections 5 and 6) or by empirical estimation using market data, the approach enables analysts to approximate the call price easily. Specifically, the call prices so estimated capture the correlation, skewness, and kurtosis of the two underlying price processes. In our implementation, we match the first two moments of the actual and approximating distributions. Our approach is similar in spirit to that of the Tankov and Ménassé (2015). There, they stated that "Our closed formula represents the indifference price as a linear combination of the Black-Scholes price and correction terms which
depend on the variance, skewness and kurtosis of the underlying Lévy process, and the derivatives of the Black-Scholes price."

2. Preliminaries

Let $Y_1$ and $Y_2$ be two random variables and $F_{Y_1,Y_2}$ be the joint distribution function. We use $A_{Y_1,Y_2}$ to denote an approximating distribution of $F_{Y_1,Y_2}$. We consider the cases when the respective density functions $f_{Y_1,Y_2}$ and $a_{Y_1,Y_2}$ exist.

Define the $(r, s)$th moment of $(Y_1, Y_2)$ about the origin $(0, 0)$ under $F_{Y_1,Y_2}$ by

$$
\mu_{r,s}(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^r y_2^s dF_{Y_1,Y_2}(y_1,y_2).
$$

It follows that $\mu_{r,0}$ is the $r$th moment of $Y_1$ and $\mu_{0,s}$ is the $s$th moment of $Y_2$. Also, $\mu_{r,s}$ is known as the $(r,s)$th product moment of $(Y_1,Y_2)$. The moments of $(Y_1,Y_2)$ are defined by the moment generating function (m.g.f)

$$
\mathbb{M}(F, t_1, t_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mathbb{E}[e^{t_1 y_1 + t_2 y_2}] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\mu_{r,s}(F) t_1^r t_2^s}{r!s!}.
$$

Similarly, we define the $(r, s)$th product moment of $(Y_1, Y_2)$ about the mean $(\mu_{10}, \mu_{02})$ under $F_{Y_1,Y_2}$ by

$$
\mu_{r,s}(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_1 - \mu_{10})^r (y_2 - \mu_{02})^s dF_{Y_1,Y_2}(y_1, y_2).
$$

In this paper, we consider the cases when these moments exist for all nonnegative integers $r$ and $s$ such that $0 < r + s \leq 4$.

Let $\Phi(F, t_1, t_2)$ denote the characteristic function (c.f.) of $F_{Y_1,Y_2}$, i.e.,

$$
\Phi(F, t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1 y_1 + it_2 y_2} dF_{Y_1,Y_2}(y_1, y_2)
$$

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where \( i^2 = -1 \). The c.f. is the Fourier transform of the density. The joint c.f. and joint m.g.f. of the two random variables are related by

\[
\Phi(F, t_1, t_2) = M(F, it_1, it_2).
\]

The bivariate cumulants \( \kappa_{r,s} \), also known as the product cumulants, of the two random variables are defined by

\[
\ln M(F, t_1, t_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\kappa_{r,s}(F)t_1^r t_2^s}{r!s!},
\]

where \( \kappa_{00} \equiv 0 \) (the term cumulant was coined by Thiele (1903)). Hence

\[
\exp \left( \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\kappa_{r,s}(F)t_1^r t_2^s}{r!s!} \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\mu_{r,s}(F)t_1^r t_2^s}{r!s!}
\]

where the right side of (2) is due to (1) (the above equation is given as (3.72) in Stuart and Ord (1994)). Since \( e^x = \sum_{n=0}^{\infty} x^n/n! \), the left side of (2) can be expanded as powers of \( t_1 \) and \( t_2 \). The product moments \( \mu_{r,s}(F) \) can be found by matching the coefficients of associated with the term \( t_1^r t_2^s \) on the right side of (2). From these relations, the product cumulants can also be expressed in terms of product moments. These relations are listed in detail in Stuart and Ord (while the latest edition was published in 2005, the original version published under the authorship of Kendall and Stuart appeared in 1943). For the cases where \( 0 < r + s \leq 4 \), we cite those results needed in subsequent development (e.g., see p. 90 and p. 107 Stuart and Ord (1994)):

\[
\begin{align*}
\kappa_{10}(F) &= \mu_{10}^1(F) & \kappa_{20}(F) &= \mu_{20}(F) & \kappa_{30}(F) &= \mu_{30}(F) \\
\kappa_{40}(F) &= \mu_{40}(F) - 3\mu_{20}^2(F) & \kappa_{11}(F) &= \mu_{11}(F) & \kappa_{21}(F) &= \mu_{21}(F) \\
\kappa_{31}(F) &= \mu_{31}(F) - 3\mu_{20} \mu_{11}(F) & \kappa_{22}(F) &= \mu_{22}(F) - \mu_{20}(F) \mu_{02}(F) - 2\mu_{11}^2(F)
\end{align*}
\]
The results for $\kappa_{01}(F)$, $\kappa_{02}(F)$, $\kappa_{03}(F)$, $\kappa_{04}(F)$, $\kappa_{12}(F)$, and $\kappa_{13}(F)$ can be found by symmetry. We also note that all the terms defined in this section are defined similarly when the approximating distribution $A_{Y_1Y_2}$ is in effect.

### 3. Bivariate Edgeworth Expansion

In this section, we approximate the true density $f_{Y_1,Y_2}$ in terms of the approximating density $a_{Y_1,Y_2}$ and their product cumulants $\kappa_{rs}$. The key result is given in Theorem 1. Our derivation of the theorem is informal but it is intuitive. It is different than that of Shiganov (1986). The assumptions about the existence of all orders of product moments and partial derivatives of density functions substantially simplify the proof. A formal and rigorous proof of the theorem for the multidimensional case can be found in Shiganov (1986). There the expansion was stated as Edgeworth-Cramér expansion.

**Theorem 1.** Assume that

$$
\frac{d^{r+s}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1,y_2)
$$

exist for nonnegative integers $r$ and $s$ such that $r + s \geq 0$. The series expansion for $f_{Y_1,Y_2}$ in terms of $a_{Y_1,Y_2}$ and their product cumulants $\kappa_{rs}$ is given by

$$
f_{Y_1,Y_2}(y_1,y_2) = a_{Y_1,Y_2}(y_1,y_2) + \sum_{r+s>0} \frac{E_{r,s}}{r!s!} \frac{d^{r+s}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1,y_2). \quad (3)
$$

**Proof.** See Appendix A. \qed

In our application of Edgeworth expansion for pricing spread options, we are only interested in including product moments of order $0 < r + s \leq 4$. This choice captures correlation, skewness, and kurtosis of the two price dynamics.
The following lemma prescribes the approximation formula and expresses the correction factors $E_{rs}$ in terms of product cumulants.

**Lemma 1.** In pricing a spread option we only include product cumulants $\kappa_{rs}$ such that $0 < r + s \leq 4$. In such a case, we approximate the actual price distribution $f_{Y_1, Y_2}$ using the approximating price distribution $a_{Y_1, Y_2}$ with

$$f_{Y_1, Y_2}(y_1, y_2) \approx a_{Y_1, Y_2}(y_1, y_2) + \sum_{0 < r + s \leq 4} E_{rs} \frac{(-1)^{r+s}}{r!s!} \frac{d^{r+s}}{dy_1^r dy_2^s} a_{Y_1, Y_2}(y_1, y_2).$$

Moreover, the correction factors $E_{rs}$ are related to product cumulants. Their relations are specified in Appendix B.

**Proof.** See Appendix B. \qed

In the spirit of Jarrow and Rudd (1982), we use a bivariate lognormal as the approximating price distribution at maturity. As $Y_i = \log(S_i), \ i = 1, 2$, $(Y_1, Y_2) \sim BVN((\mu_1, \mu_2), (\rho, \sigma_1, \sigma_2))$. Since $BVN$ is completely characterized by the last five parameters and the product cumulants beyond order two of $BVN$ vanish (e.g., Holmquist (1988) simplifies the approximation formula (4) further).

**Lemma 2.** If we set $\kappa_{0i}(F) = \kappa_{0i}(A)$ and $\kappa_{i0}(F) = \kappa_{i0}(A)$ for $i = 1$ and
2, and \( \rho(F) = \rho(A) \), then (4) reduces to

\[
\begin{align*}
& f_{Y_1,Y_2}(y_1, y_2) \approx a_{Y_1,Y_2}(y_1, y_2) \\
& \quad - \frac{\kappa_{12}(F)}{2!} \frac{d^3}{dy_1 dy_2^2} a_{Y_1,Y_2}(y_1, y_2) - \frac{\kappa_{21}(F)}{2!} \frac{d^3}{dy_1^2 dy_2} a_{Y_1,Y_2}(y_1, y_2) \\
& \quad - \frac{\kappa_{30}(F)}{3!} \frac{d^3}{dy_1^3} a_{Y_1,Y_2}(y_1, y_2) - \frac{\kappa_{03}(F)}{3!} \frac{d^3}{dy_2^3} a_{Y_1,Y_2}(y_1, y_2) \\
& \quad + \frac{\kappa_{13}(F)}{3!} \frac{d^4}{dy_1 dy_2^2} a_{Y_1,Y_2}(y_1, y_2) + \frac{\kappa_{31}(F)}{3!} \frac{d^4}{dy_1^3 dy_2} a_{Y_1,Y_2}(y_1, y_2) \\
& \quad + \frac{\kappa_{40}(F)}{4!} \frac{d^4}{dy_1^4} a_{Y_1,Y_2}(y_1, y_2) + \frac{\kappa_{04}(F)}{4!} \frac{d^4}{dy_2^4} a_{Y_1,Y_2}(y_1, y_2) \\
& \quad + \frac{\kappa_{22}(F)}{2! 2!} \frac{d^4}{dy_1^2 dy_2^2} a_{Y_1,Y_2}(y_1, y_2).
\end{align*}
\]

Proof. Using \( E_{ij} = \kappa_{ij}(F) \) for \( i + j = 3 \) and \( 4 \), and 0 otherwise, and the assumptions of the lemma, (5) follows. \( \square \)

Lemma 2 shows that if we match the means, variances, and correlations of the true distribution \( F \) and approximating distribution \( A \), then the difference between the two distributions can be approximated by a series expansion on the right side of (5) involving higher order cumulants of the two distributions and the derivatives of the approximating density \( a_{Y_1,Y_2} \).
4. Pricing Spread Options

We now demonstrate the application of bivariate Edgeworth expansion to the pricing of spread option. Let $S_1$ and $S_2$ denote the price processes of the two underlying assets. Under the risk-neutral measure $Q$, the price of the call is given by

$$c(F) = e^{-rT}E^Q [(S_2(T) - S_1(T) - K)^+] = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_2 - s_1 - K)^+ f_{S_1, S_2}(s_1, s_2) ds_1 ds_2$$

where $f_{S_1, S_2}$ denote the joint density of $S_1$ and $S_2$ under $Q$. In the context of Edgeworth expansion, $f_{S_1, S_2}$ denotes the true density. As before, we let $a_{S_1, S_2}$ denote the approximating bivariate density.

Substituting the Edgeworth expansion (4) into (6), we obtain

$$c(F) \approx c(A) + e^{-rT} \sum_{0<r+s\leq 4} E_{rs} \frac{(-1)^{r+s}}{r!s!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_2 - s_1 - K)^+ \times \frac{d^{r+s}}{ds_1^r ds_2^s} a_{S_1, S_2}(s_1, s_2) ds_1 ds_2$$

(7)

where

$$c(A) = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_2 - s_1 - K)^+ a_{S_1, S_2}(s_1, s_2) ds_1 ds_2.$$  

(8)

When the conditions of Theorem 1 are met, the price of a spread option whose underlying price processes follow arbitrary distributions can be approximated by the Edgeworth expansion (7) with any arbitrary chosen approximating price distributions. As many option valuation results pertaining to lognormal price distributions are known, in what follows we choose the geometric Brownian motion paradigm for evaluating $c(A)$. Lemma 2 implies that the right side of (7) can be decomposed into three parts, the price of the spread option under...
GBM, an adjustment term involving skewness (terms with \( r + s = 3 \)), and an adjustment term involving skewness (terms with \( r + s = 4 \)). We refer them as Terms 1, 2, and 3, respectively.

4.1 Pricing Spread Options under GBM

Let \( Y_i = \log S_i(T), i = 1, 2 \). We assume \((Y_1, Y_2) \sim BVN((\mu_1, \mu_2), (\rho, \sigma_1, \sigma_2))\).

We use (8) to write

\[
\exp(-rT) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2
\]

where \( a_{Y_1,Y_2} \) follows the bivariate normal density. We now follow Pearson’s (1995) approach to reduce the double integration to a single integration. Our derivation differs from that used by Pearson but after some algebraic manipulations it would produce the identical result.

For simplicity, we state the double integral in (9) as

\[
IntA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2
\]

\[
= \int_{-\infty}^{\infty} G_{Y_1}(y_1) a_{Y_1}(y_1) dy_1
\]

where we define

\[
G_{Y_1}(y_1) = \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_2|Y_1}(y_2|y_1) dy_2.
\]

The marginal density for \( Y_1 \) is \( N(\mu_1, \sigma_1) \) and the conditional density is given by

\[
a_{Y_2|Y_1}(y_2|y_1) = \frac{1}{\sqrt{2\pi \sigma_2^2}} \exp \left\{ -\frac{1}{2\sigma_2^2} [y_2 - M_2(y_1)]^2 \right\}
\]

(10)
where
\[\sigma_2^2 \equiv \sigma_2^2(1 - \rho^2) \quad \sigma_2^*(y_1) = \frac{\sigma_2 \rho(y_1 - \mu_1)}{\sigma_1} \quad M_2(y_1) \equiv \mu_2 + \sigma_2^*(y_1).\]

The following theorem gives a closed form expression of \(G_{Y_1}\) and hence eliminates the need to perform a numerical integration.

**Theorem 2.** We have
\[G_{Y_1}(y_1) = e^{A_2(y_1)} N(x_1(y_1)) - (e^{y_1} + K)N(x_2(y_1)) \quad (11)\]

where
\[A_2(y_1) = M_2(y_1) + \frac{\sigma_2^2}{2},\]
\[x_1(y_1) = x_2(y_1) + \sigma_2,\]
\[x_2(y_1) = \frac{M_2(y_1) - \ln(e^{y_1} + K)}{\sigma_2} \]

where \(N(\cdot)\) is the distribution function of \(N(0,1)\).

**Proof.** See Appendix C. \(\square\)

We see that Term 1 for approximating the time-0 call price of the spread option \(c(F)\), which appears on the right side of (7), can be found from
\[c(A) = e^{-rT} \int_{-\infty}^{\infty} G_{Y_1}(y_1)a_{Y_1}(y_1)dy_1.\]

As a matter of fact, Term 1 can also be computed in closed form using the formulas derived in Li (2008) and Li, Deng, and Zhou (2008).

### 4.2 Adjustment for Skewness

To adjust for the discrepancy caused by the difference in skewness between
the uses of actual and approximating distributions in evaluating a call price, we include the following (terms relating to $E_{rs}$, where $r + s = 3$)

\[
e^{-rT} \sum_{r+s=3} E_{rs} \frac{(-1)^{r+s}}{r!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_2 - y_1 - K)^{r+s} dy_1 dy_2
\]

\[
e^{-rT} \left[ \frac{\kappa_{03}(F)}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_2 - y_1 - K)^{r+s} dy_1 dy_2 + \frac{\kappa_{12}(F)}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_2 - y_1 - K)^{r+s} dy_1 dy_2 + \frac{\kappa_{30}(F)}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_2 - y_1 - K)^{r+s} dy_1 dy_2 \right]
\]

After multiple differentiations of the approximating bivariate normal density and performing the conditioning similar to those shown earlier, Term 2 can be found by single integration using

\[
-e^{-rT} \left[ q_1 \int_{-\infty}^{\infty} (y_1 - \mu_1) f_{Y_1}(y_1) G_{Y_1}(y_1) dy_1 + q_2 \int_{-\infty}^{\infty} (y_2 - \mu_2) f_{Y_2}(y_2) G_{Y_2}(y_2) dy_2 + q_3 \int_{-\infty}^{\infty} (y_1 - \mu_1)^2 f_{Y_1}(y_1) G_{Y_1}(y_1) dy_1 + q_4 \int_{-\infty}^{\infty} (y_2 - \mu_2)^2 f_{Y_2}(y_2) G_{Y_2}(y_2) dy_2 + q_5 \int_{-\infty}^{\infty} (y_1 - \mu_1)^3 f_{Y_1}(y_1) H_{Y_1}(y_1) dy_1 + q_6 \int_{-\infty}^{\infty} (y_2 - \mu_2)^3 f_{Y_2}(y_2) H_{Y_2}(y_2) dy_2 \right]
\]

where $G_{Y_1}(y_1)$ is given by (11) and $G_{Y_2}(y_2)$ is similarly defined, and $q_i$, $i = 1, ..., 6$, and $H_{Y_1}$, $i = 1, 2$ are given in Appendix D.

4.3. Adjustment for Kurtosis

To adjust for the discrepancy caused by the difference in kurtosis between the uses of actual and approximating distributions in evaluating a call price, we add the following (terms relating to $E_{rs}$, where $r + s = 4$)
\[
e^{-rT} \sum_{r+s=1}^{r+s} E_{r,s} \left( \frac{(-1)^{r+s}}{r!s!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^{r+s}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 \right)
\]

\[
e^{-rT} \left[ \frac{\kappa_{04}(F)}{4!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^4}{dy_1^4 dy_2^4} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 
+ \frac{\kappa_{12}(F)}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^2}{dy_1^2 dy_2^2} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 
+ \frac{\kappa_{21}(F)}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^4}{dy_1^2 dy_2^4} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 
+ \frac{\kappa_{31}(F)}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^4}{dy_1^4 dy_2^2} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 
+ \frac{\kappa_{40}(F)}{4!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^4}{dy_1^4 dy_2^4} a_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 \right]
\]

After multiple differentiations of the approximating bivariate normal density and performing the conditioning similar to those shown earlier, Term 3 can be found by single integration using

\[
e^{-rT} \left[ u_1 \int_{-\infty}^{\infty} G_1(y_1) a_{Y_1}(y_1) dy_1 + u_2 \int_{-\infty}^{\infty} (y_1 - \mu_1)^2 G_1(y_1) a_{Y_1}(y_1) dy_1 
+ u_3 \int_{-\infty}^{\infty} (y_1 - \mu_1) H_{Y_1}(y_1) a_{Y_1}(y_1) dy_1 + u_4 \int_{-\infty}^{\infty} (y_2 - \mu_2)^2 G_2(y_2) a_{Y_2}(y_2) dy_2 
+ u_5 \int_{-\infty}^{\infty} (y_1 - \mu_1)^2 J_{Y_2}(y_1) a_{Y_2}(y_1) dy_1 + u_6 \int_{-\infty}^{\infty} (y_1 - \mu_1)^3 H_{Y_1}(y_1) a_{Y_1}(y_1) dy_1 
+ u_7 \int_{-\infty}^{\infty} (y_2 - \mu_2)^3 H_{Y_2}(y_2) a_{Y_2}(y_2) dy_2 + u_8 \int_{-\infty}^{\infty} (y_2 - \mu_2)^3 G_2(y_2) a_{Y_2}(y_2) dy_2 
+ u_9 \int_{-\infty}^{\infty} (y_1 - \mu_1)^4 J_{Y_1}(y_1) a_{Y_1}(y_1) dy_1 \right]
\]

where, as before, \( G_{Y_i} \) and \( H_{Y_i} \), \( i = 1, 2 \) are defined earlier, and \( u_i, i = 1, \ldots, 9 \), and \( J_{Y_3}(y_1) \) are given in Appendix E.

5. Spread Options under Merton’s Jump Diffusion

Consider a spread option involving two underlying assets whose price processes
follow Merton’s jump diffusion paradigm [21]. Under risk neutral measure, the two price processes are governed by the following two stochastic differential equations

\[ dS_i(t) = (r - q_i - \lambda_i E[e^{J_i} - 1]) S_i(t) dt + \sigma_i S_i(t) dW_i(t) + (e^{J_i} - 1) S_i(t) dN_i(t) \]

(12)

where the two jump processes \( N_i \sim \text{Poisson}(\lambda_i) \), \( i = 1 \) and \( 2 \), the two jump size distributions \( J_i \sim N(m_i, s_i^2) \), \( i = 1 \) and \( 2 \), and \( q_1 \) and \( q_2 \) are the instantaneous dividend yields, and \( r \) is the riskfree rate. For each asset, the respective price process specified by (12) is exactly that considered by Merton (1976) (also, see Jarrow and Rudd (1982), p. 359). We assume i) \( N_1 \) and \( N_2 \) are independent, ii) \( J_1 \) and \( J_2 \) are independent, iii) \( W_1 \) and \( W_2 \) are correlated with \( dW_1 dW_2 = \rho dt \), and iv) \( \{N_j\}, \{J_j\}, \) and \( \{W_j\} \) are mutually independent.

Since \( Y_i(t) = \log S_i(t) \), we apply Itô formula for the jump diffusion process and obtain

\[ dY_i(t) = \left( r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i E[e^{J_i} - 1] \right) dt + \sigma_i W_i(T) + J_i(t) dN_i(t) \quad i = 1, 2 \]

Integrating the above, we obtain

\[ Y_i(T) = Y_i(0) + \left( r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i E[e^{J_i} - 1] \right) T + \sigma_i W_i(T) + \sum_{k=1}^{N(T)} J_i(k) \]

(13)

where the last equality is due to \( J_i \sim N(m_i, s_i^2) \) and an application of the m.g.f.
of a normal density. For notational convenience, we write the above as

$$Y_i(T) = \Lambda_i(T) + L_i$$

where

$$\Lambda_i(T) := \log S_i(0) + \left( r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i \left[ e^{m_i + \frac{\sigma_i^2}{2}} \right] \right) T + \sigma_i W_i(T)$$

and

$$L_i \equiv \sum_{k=1}^{N_i(T)} J_i(k).$$

To simplify the above further, we write

$$Y_i(T) = \mu_i + \Omega_i Z_i + L_i$$

where

$$\mu_i : = \log S_i(0) + \left( r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i \left[ e^{m_i + \frac{\sigma_i^2}{2}} \right] \right) T$$

$$\Omega_i : = \sigma_i \sqrt{T}$$

and $Z \sim N(0,1)$.

To apply the Edgeworth Expansion for pricing the spread option, we assume that the jump diffusion processes (13) generates the true distribution $F_{Y_1,Y_2}$ at maturity. Since the valuation of the spread option is done by (5), what remain to be determined are the product cumulants $\kappa_{rs}(F)$, where $r + s = 3$ and 4.

5.1 Moments of $(L_1, L_2)$ under $F$

As $L_i$ is a compound Poisson process, its m.g.f. is given by $E[e^{tL_i}] = \exp \{ \lambda_i T (M_{\lambda_i}(t) - 1) \}$ (e.g., see Shreve (2000), p. 471), where $M_{\lambda_i}(t)$ is the
m.g.f. of jump size $J_i$. Since $J_i \sim N(m_i, s_i^2)$, we have

$$M_{J_i}(t) = \exp \left\{ m_i t + \frac{s_i^2 t}{2} \right\}.$$  

This gives the m.g.f. of $L_i$

$$E[e^{tL_i}] = \exp \left\{ \lambda_i T \left( \exp \left( m_i t + \frac{s_i^2 t}{2} \right) - 1 \right) \right\}$$

Expanding the above yields the first four moments of $L_i$

$$E[L_i] = m_i \lambda_i T$$
$$E[L_i^2] = (m_i^2 + s_i^2) \lambda_i T + m_i^2 \lambda_i^2 T^2$$
$$E[L_i^3] = (m_i^3 + 3m_i s_i^2) \lambda_i T + (3m_i^3 + 3m_i s_i^2) \lambda_i^2 T^2 + m_i^3 \lambda_i^3 T^3$$
$$E[L_i^4] = (m_i^4 + 6m_i^2 s_i^2 + 3s_i^4) \lambda_i T + (7m_i^4 + 18m_i^2 s_i^2 + 3s_i^2) \lambda_i^3 T^2 + (6m_i^4 + 6m_i^2 s_i^2) \lambda_i^2 T^3 + m_i^4 \lambda_i^4 T^4.$$  

5.2 Product Moments of $(Y_1, Y_2)$ under $F$

The first two moments of $Y_i$ are

$$E[Y_i] \equiv M_i = \mu_i + \lambda_i m_i T \quad Var[Y_i^2] \equiv \Omega_i^2 = \Omega_i^2 + (m_i^2 + s_i^2) \lambda_i T$$

and hence $Y_i - E[Y_i] = \Omega_i Z_i + L_i - \lambda_i m_i T$. The $(r, s)$th product moment of $(Y_1, Y_2)$ is defined by

$$\mu_{rs}(F) = E \left[ (\Omega_1 Z_1 + (L_1 - \lambda_1 m_1 T))^r \ (\Omega_2 Z_2 + (L_2 - \lambda_2 m_2 T))^s \right].$$

where $r$ and $s$ are nonnegative integers. Recall that for a standard bivariate
normal distribution, for $0 < r + s \leq 4$, we have

\begin{align*}
E[Z_1^2 Z_2^0] &= 1 & E[Z_1^0 Z_2^2] &= 3 \\
E[Z_1 Z_2] &= \rho & E[Z_1^2 Z_2^2] &= 1 + 2\rho^2 \\
E[Z_1 Z_3^2] &= 3\rho,
\end{align*}

$E[Z_1^0 Z_2^2]$, $E[Z_1^2 Z_2^2]$, and $E[Z_1^2 Z_2^2]$ can be stated by symmetry, and 0 otherwise. Using (14) and (15), and by invoking the independence assumptions made at the outset of the section, we find the product moments for $3 \leq r + s \leq 4$ below:

\begin{align*}
\mu_{21}(F) &= 0 & \mu_{30}(F) &= (m_1^3 + 3m_1s_1^2)\lambda_1T \\
\mu_{40}(F) &= 3\Omega_1^4 + (m_1^4 + 6m_1^2s_1^2 + 3s_1^4 + 6\Omega_1^2(m_1^2 + s_1^2))\lambda_1T + 3(m_1^2 + s_1^2)^2\lambda_1^2T^2 \\
\mu_{22}(F) &= \Omega_1^2\Omega_2^2(1 + 2\rho^2) + \Omega_1^2(m_1^2 + s_1^2)\lambda_1T + \Omega_1^2(m_2^2 + s_2^2)\lambda_2T \\
&\quad + (m_1^2 + s_1^2)(m_2^2 + s_2^2)\lambda_1\lambda_2T^2 \\
\mu_{31}(F) &= 3\rho\Omega_1^4\Omega_2 + 3\rho\Omega_1\Omega_2(s_1^2 + m_1^2)\lambda_1T
\end{align*}

where $\mu_{12}(F)$, $\mu_{33}(F)$, $\mu_{34}(F)$, and $\mu_{41}(F)$ can be found by symmetry.

5.3 Product Cumulants of $(Y_1, Y_2)$ under $F$

Using the results shown in Section 2, we find the product cumulants of $(Y_1, Y_2)$ under $F$ below:

\begin{align*}
\kappa_{21}(F) &= \mu_{21}(F) = 0 \\
\kappa_{30}(F) &= \mu_{30}(F) = (m_1^3 + 3m_1s_1^2)\lambda_1T \\
\kappa_{31}(F) &= \mu_{31}(F) - 3\mu_{20}(F)\mu_{11}(F) = 0 \\
\kappa_{22}(F) &= \mu_{22}(F) - \mu_{20}(F)\mu_{02}(F) - 2\mu_{11}(F) = 0 \\
\kappa_{40}(F) &= \mu_{40}(F) - 3\mu_{20}^2(F) = (m_1^4 + 6m_1^2s_1^2 + 3s_1^4)\lambda_1T
\end{align*}
where $\kappa_{03}(F)$, $\kappa_{13}(F)$, and $\kappa_{04}(F)$ can be found by symmetry.

5.4 A Numerical Example

Consider a numerical example with the follow parameters:

$r = 0.1$, $T = 1$, $\rho = 0.5$, and $K = 2$.

For the above example, we use Matlab to compute the call price of the spread option. The results are summarized below:

<table>
<thead>
<tr>
<th>Term</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term1 (the GBM solution)</td>
<td>15.3437</td>
</tr>
<tr>
<td>Term 2 (adjustment for skewness)</td>
<td>0.7450</td>
</tr>
<tr>
<td>Term 3 (adjustment for kurtosis)</td>
<td>-1.0463</td>
</tr>
<tr>
<td>Approximated call value</td>
<td>15.0424</td>
</tr>
</tbody>
</table>

In a recent paper, Jackson, Jaimungal, and Surkov (2008/2009) proposed a Fourier transform based algorithm for solving option pricing problems under the Lévy paradigm. Using their approach to solve the above problem, we find the call price is 15.0292. The Edgeworth based approach presented in this paper enables us to assess the relatively contributions made by the skewness
and kurtosis of the underlying price processes. Finally, we remark that a Monte Carlo simulation using 200,000 samples produced a solution of 15.071.

5.5 Computations Based on Simulated Prices at Maturity

For applications, one may know the underlying stochastic differentiation equations governing the price processes and/or their parameters. If the first four moments of the price process can be assessed empirically, then the approach proposed in the paper can be useful in coming up with approximating prices. To explore this point further, we use the parameters specified in the last subsection to generate random sample of $S_1(T)$ and $S_2(T)$ of size 100,000. Based of this data set, we computed the sample estimates of the first four moments as the input to the Edgeworth-based method. With a model specification of $(K, r, T) = (2, 0.1, 1)$, we find the calculated values as follows

<table>
<thead>
<tr>
<th>Term</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term 1 (the GBM solution)</td>
<td>15.2760</td>
</tr>
<tr>
<td>Term 2 (adjustment for skewness)</td>
<td>0.6890</td>
</tr>
<tr>
<td>Term 3 (adjustment for kurtosis)</td>
<td>-1.0087</td>
</tr>
<tr>
<td>Approximated call value</td>
<td>14.9563</td>
</tr>
</tbody>
</table>

As expected, the results differ only slightly from those reported in the last subsection. It is important to note that in this case the model does not "know" that it is Merton's jump diffusion that generated the prices at maturity. It only uses the price data at maturity.

If traders can speculate the magnitudes of the first four moments of the price processes, we see the approach provides a nice way the produce the prices and their components. For quantitative analysts proposing models that are only solvable by Monte Carlo simulations or intricate approximating algorithms, they can obtain empirically estimated first four moments and use the Edge-
worth approach to cross-validate their results. This is a very nice feature of the Edgeworth-based approach.

6. Spread Options under Kou’s Double Jump Diffusion

Spread options are commonly used in energy and power industries to hedge fluctuations of prices of two correlated commodities, e.g., prices of electricity and coal. It is well known that price volatilities in power and energy markets are substantially higher than that observed in the equity and fixed-income markets. Moreover, there the phenomena known as asymmetric leptokurtic feature and volatility smile are more pronounced. Kou (2002) proposed the use of a double exponential jump diffusion model to mitigate these problems. Under the paradigm, we are dealing with a version of Lévy processes (e.g., see p. 112 of Cont and Tankov (2004)). In this section, we will use this paradigm to model price movements of the two underlying assets of a spread option.

6.1 The Kou’s Double Jump Diffusion

For Asset $i$, $i = 1, 2$, the SDEs governing the price processes are identical to that of (12). Under Kou’s assumption, $J_i$ follows an asymmetric double exponential distribution, i.e., the density of $J_i$ is given by

$$f_{J_i}(y) = p_i \cdot \eta_{i1} e^{-\eta_{i1} y} 1_{\{y \leq 0\}} + q_i \cdot \eta_{i2} e^{-\eta_{i2} y} 1_{\{y < 0\}} \quad i = 1, 2$$

where $\eta_{i1} > 1$, $\eta_{i2} > 0$, $p_i \geq 0$, $q_i \geq 0$, and $p_i + q_i = 1$, for $i = 1, 2$. In this construct, we have

$$J_i \overset{d}{=} \begin{cases} \xi_i^+ & \text{with probability } p_i \\ -\xi_i^- & \text{with probability } q_i \end{cases}$$

where $\xi_i^+ \sim \exp(\eta_{i1})$ and $\xi_i^- \sim \exp(\eta_{i2})$, with respective means $1/\eta_{i1}$ and $1/\eta_{i2}$. 

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We see
\[ E[J_i] = \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \quad i = 1, 2 \]
and for \( i = 1, 2 \), we have
\[ E[e^{J_i}] = \frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1} \quad \eta_{i1} > 1, \ \eta_{i2} > 0. \]

Using (22) and applying Itô formula, the risk-neutral log price (i.e., the return) of Asset \( i \) at time \( T \) can be found. It is given by
\[
Y_i(T) = Y_i(0) + \left[ r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i \left( \frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1} - 1 \right) \right] T \\
+ \sigma_i W_i(T) + \sum_{k=1}^{N_i(T)} J_i(k)
\]
(e.g., see Kou (2002)). For notational simplicity, for Asset \( i \), we define
\[
Y_i(T) := \Lambda_i(T) + L_i := \mu_i + \Omega_i Z_i + L_i
\]
where
\[
\mu_i := \log S_i(0) + \left[ r - q_i - \frac{1}{2} \sigma_i^2 - \lambda_i \left( \frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1} - 1 \right) \right] T \\
\Omega_i := \sigma_i \sqrt{T} \\
\Lambda_i(T) := \mu_i + \Omega_i Z_i \\
L_i := \sum_{k=1}^{N_i(T)} J_i(k)
\]
and \( Z_i \sim N(0, 1) \).

6.2 Moments of \((Y_1, Y_2)\) under \( F \)

In order to derive product cumulants and the correction factors under Kou’s model, we first need to find the m.g.f. of jump size. For Asset \( i \), we see the
m.g.f. can be derived as follows:

\[ M_{J_t}(t) = E\left[e^{tJ_t}\right] \]

\[ = \int_{-\infty}^{\infty} e^{tx} \left( p_i \eta_{t1} e^{-\eta_{t1}x} 1_{\{x \geq 0\}} + e^{tx} q_i \eta_{t2} e^{\eta_{t2}x} 1_{\{x < 0\}} \right) dx \]

\[ = \int_{-\infty}^{0} e^{tx} q_i \eta_{t2} e^{\eta_{t2}x} dx + \int_{0}^{\infty} e^{tx} p_i \eta_{t1} e^{-\eta_{t1}x} dx \]

\[ = q_i \eta_{t2} \int_{-\infty}^{0} e^{(t+\eta_{t2})x} dx + p_i \eta_{t1} \int_{0}^{\infty} e^{-(t+\eta_{t1})x} dx \]

\[ = \frac{q_i \eta_{t2}}{t + \eta_{t2}} - \frac{p_i \eta_{t1}}{t - \eta_{t1}} \quad t + \eta_{t2} \geq 0 \quad \text{and} \quad t - \eta_{t2} \leq 0 \]

or \(-\eta_{t2} \leq t \leq \eta_{t1}\). Using series expansion on \( t \), the above m.g.f. can be stated as

\[ M_{J_t}(t) = 1 + \sum_{j=1}^{\infty} \left( \frac{p_i}{\eta_{t1}} + (-1)^j \frac{q_i}{\eta_{t2}} \right) t^j. \]

By definition of the m.g.f., we have

\[ E\left[e^{tL_i}\right] = 1 + t m_{i1} + \frac{t^2}{2} m_{i2} + \frac{t^3}{3!} m_{i3} + \frac{t^4}{4!} m_{i4} + \cdots \]

where \( m_{ij} \) is the \( j \)th moment of \( L_i \). Since \( L_i \) is a compound Poisson process, we apply (11.3.2) of Shreve (2000, p. 471) and write the m.g.f. of \( L_i \) as

\[ E\left[e^{tL_i}\right] = \exp \{ \lambda_i T (M_{J_t}(t) - 1) \} \]

\[ = \exp \left\{ \lambda_i T \sum_{j=1}^{\infty} \left( \frac{p_i}{\eta_{t1}} + (-1)^j \frac{q_i}{\eta_{t2}} \right) t^j \right\} \]

We expand (17) as a polynomial in \( t \) and match the respective coefficients with those in (16). This yields the first four moments of \( L_i \). They are
\[ m_{i1} = \lambda_i T \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right) \]

\[ m_{i2} = 2\lambda_i T \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right) + (\lambda_i T)^2 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right)^2 \]

\[ m_{i3} = 6\lambda_i T \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right) + 6(\lambda_i T)^2 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right) \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right) + (\lambda_i T)^3 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right)^3 \]

\[ m_{i4} = 24\lambda_i T \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right) + 12(\lambda_i T)^2 \left[ 2 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right) \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right) + \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right)^2 \right] \]

\[ + 12(\lambda_i T)^3 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right)^2 \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right) + (\lambda_i T)^4 \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right)^4 \]

We are now ready to derive product moments of \((Y_1, Y_2)\) under \(F\). Recall

\[ Y_i(T) = \mu_i + \Omega_i Z_i + L_i. \]

We take expectation of the above and find

\[ E_i[Y_i(T)] = \mu_i + \Omega_i E[Z_i] + E[L_i] = \mu_i + \lambda_i T \left( \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}} \right). \]

Now

\[ Y_i(T) - E[Y_i(T)] = (\mu_i + \Omega_i Z_i + L_i) - (\mu_i + E[L_i]) = \Omega_i Z_i + L_i - E[L_i] = \Omega_i Z_i + L_i - m_{i1} \]

and therefore

\[ Var[Y_i(T)] = E \left[ (\Omega_i Z_i + L_i - m_{i1})^2 \right] \]

\[ = \Omega_i^2 + E \left[ L_i^2 \right] - m_{i1}^2 \]

\[ = \Omega_i^2 + m_{i2} - m_{i1}^2 \]

\[ = \Omega_i^2 + 2\lambda_i T \left( \frac{p_i}{\eta_{i1}} + \frac{q_i}{\eta_{i2}} \right) \]

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Therefore the product moment $\mu_{11}(F)$ of $(Y_1, Y_2)$ under $F$ is given by

$$\mu_{11}(F) = E \left[ \prod_{i=1}^{2} (\Omega_i Z_i + L_i - m_{11}) \right] = \rho \Omega_1 \Omega_2$$

The higher order product moments are

$$\mu_{21}(F) = 0$$
$$\mu_{30}(F) = 6\lambda_1 T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)$$
$$\mu_{40}(F) = 3\Omega_1^2 + 12\lambda_1 T \left[ 2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + \Omega_1^2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \right] + 12 (\lambda_1 T)^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right)^2 - (\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4$$
$$\mu_{22}(F) = (1 + 2\rho^2) (\Omega_1^2 \Omega_2^2) + \Omega_1^2 + (2\lambda_2 T) \left( \frac{p_2}{\eta_{21}} + \frac{q_2}{\eta_{22}} \right) + \Omega_2^2 \cdot (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + 4\lambda_1 \lambda_2 T^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) \left( \frac{p_2}{\eta_{21}} + \frac{q_2}{\eta_{22}} \right)$$
$$\mu_{31}(F) = 3\rho \Omega_1^3 \Omega_2 + 3\rho \Omega_1 \Omega_2 \cdot (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right)$$

In Appendix $F$, we give their derivations. The product cumulants $\kappa_{rs}$, $r + s = 3$ and 4, under the true distribution $F$ are found accordingly. They are

$$\kappa_{11}(F) = \mu_{11}(F) = \rho \Omega_1 \Omega_2$$
$$\kappa_{21}(F) = \mu_{21}(F) = 0$$
$$\kappa_{30}(F) = \mu_{30}(F) = 6\lambda_1 T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)$$
$$\kappa_{31}(F) = \mu_{31}(F) - 3\mu_{20}(F) \mu_{11}(F)$$
$$\quad = 3\rho \Omega_1^3 \Omega_2 + 3\rho \Omega_1 \Omega_2 \cdot (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) - 3\rho \Omega_1 \Omega_2 \left[ \Omega_1^2 + 2\lambda_1 T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) \right] = 0$$
$$\kappa_{22}(F) = \mu_{22}(F) - \mu_{20}(F) \mu_{02}(F) - 2\mu_{11}(F) = 0$$
$$\kappa_{40}(F) = \mu_{40}(F) - 3\mu_{20}(F)^2 = 24\lambda_1 T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) - (\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4$$
and finally we obtain the nonzero correction factors

\[ E_{30} = \kappa_{30}(F) = 6\lambda_1 T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \]

\[ E_{40} = \kappa_{40}(F) = 24\lambda_1 T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) - (\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4 \]

These results form the input for application of the Edgeworth expansion.

6.3 A Numerical Example

Consider a numerical example with the following model specifications

<table>
<thead>
<tr>
<th></th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_i(0) )</td>
<td>100</td>
<td>97</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>0.16</td>
<td>0.10</td>
</tr>
<tr>
<td>( \lambda_i )</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>( p_i )</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>( \eta_{1i} )</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>( \eta_{2i} )</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

In addition, we assume \( r = 0.05 \), \( \rho = 0.05 \), \( T = 4 \), and \( K = 4 \).

For the above example, we use Matlab to compute the call price of the spread option. The results are summarized below:

<table>
<thead>
<tr>
<th></th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term1 (the GBM solution)</td>
<td>7.4814</td>
</tr>
<tr>
<td>Term 2 (adjustment for skewness)</td>
<td>-0.9095</td>
</tr>
<tr>
<td>Term 3 (adjustment for kurtosis)</td>
<td>-1.3511</td>
</tr>
<tr>
<td>Approximated call value</td>
<td>5.2207</td>
</tr>
</tbody>
</table>
The above figures show that the effects of skewness and kurtosis are relatively large. If we had used the two-dimensional GBM to price the call, the call price would have been 7.4814. The two higher moments bring it down to 5.2207. Here, we demonstrated that the Edgeworth based approach presented in this paper enables us to assess the relatively contributions made by the skewness and kurtosis of the underlying return processes. We mention in passing, the computation took 1.19 seconds. A Monte Carlo simulation using 300,000 samples produced a solution of 5.6453. This simulation took 222.49 seconds.

6.4 Computations Based on Empirically Observed Prices at Maturity

In applications, one may make assumptions about the underlying stochastic differentiation equations governing the price processes and/or their parameters. If the first four moments of the price process can be assessed empirically, then the approach proposed in the paper can be useful in coming up with approximating prices. To explore this point further, we use the parameters specified in the last subsection to generate random sample of $S_1(T)$ and $S_2(T)$ of size 300,000. Based of this data set, we computed the sample estimates of the first four moments as the input to the Edgeworth-based method. With a model specification of $(K, r, T) = (4, 0.05, 0.5)$, we find the calculated values as follows

<table>
<thead>
<tr>
<th>Term</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term1 (the GBM solution)</td>
<td>7.4834</td>
</tr>
<tr>
<td>Term 2 (adjustment for skewness)</td>
<td>-0.9402</td>
</tr>
<tr>
<td>Term 3 (adjustment for kurtosis)</td>
<td>-1.3509</td>
</tr>
<tr>
<td>Approximated call value</td>
<td>5.1923</td>
</tr>
</tbody>
</table>

As expected, the results differ only slightly from those reported earlier. It is important to note that in this case the model does not "know" that it is Kou's
double exponential jump diffusion model that generated the prices at maturity. It only uses the price data at maturity.

If traders can speculate the magnitudes of the first four moments of the price processes, we see the approach provides a nice way to produce the prices and their components. For quantitative analysts proposing models that are only solvable by Monte Carlo simulations or intricate approximating algorithms, they can obtained empirically estimated first four moments and use the Edgeworth approach to cross-validate their results. This is a very nice feature of the Edgeworth-based approach.

7. Conclusion

This paper proposes a bivariate generalized Edgeworth expansion for pricing spread option. This paper is a generalization of Jarrow and Rudd (1992) who introduced the idea of using univariate Edgeworth expansion for pricing call options when the underling price process deviates from the lognormal distribution. As in Jarrow and Rudd, we see that when the first four product cumulants are known, the expansion yields adequate approximations for problems that do not have closed-form solutions. In theory, the higher orders of differentiation and integration can be problematic for numerical implementation. The paper demonstrates that after proper manipulations, the computation simplifies to integrations involving single integrals. Hence it significantly reduces the cost of computation and potentials for computation errors. More importantly, the option price obtained by Edgeworth expansion provides a valuable insight about its constituents - namely, the relative contributions made by the various moments of the asset prices (e.g., correlation, skewness, and kurtosis). This advantage is not shared by methods such as numerical solutions of PDE or Monte Carlo simulation. For financial analysts with real-world applications in mind, using
market data, one can estimate empirical product cumulants (e.g., see Stuart and Ord (1994)) and proceed with the aforementioned approximation. In summary, the approach described in this paper would provide a viable alternative to the solution of the pricing problem.

References


Duan, J. C., Gauthier G., and Simonato, J., An analytical approximation for


Shiganov, I. S., A multidimensional counterpart of the Gram-Charlier and Edgeworth-Cramér Expansions. *Journal of Mathematical Sciences*, 1986, 32,


Appendix A. Proof of Theorem 1.

\[ \Phi(F, t_1, t_2) = M(F, it_1, it_2), \]

we use (2) to write

\[ \ln \Phi(F, t_1, t_2) = \sum_{r+s \geq 0} \kappa_{r,s}(F) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \]  
\[ \Phi(F, t_1, t_2) = M(F, it_1, it_2), \]

and

\[ \ln \Phi(A, t_1, t_2) = \sum_{r+s \geq 0} \kappa_{r,s}(A) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \]  
\[ \Phi(A, t_1, t_2) = M(A, it_1, it_2), \]
Subtracting (A2) from (A1), we find

\[ \ln \Phi(F, t_1, t_2) = \ln \Phi(A, t_1, t_2) + \sum_{r+s \geq 0} (\kappa_{r,s}(F) - \kappa_{r,s}(A)) \frac{(it_1)^r (it_2)^s}{r! s!} \]

Hence, we have

\[ \Phi(F, t_1, t_2) = \exp \left( \sum_{r+s \geq 0} (\kappa_{r,s}(F) - \kappa_{r,s}(A)) \frac{(it_1)^r (it_2)^s}{r! s!} \right) \Phi(A, t_1, t_2) \quad (A3) \]

The bivariate inverse Fourier transforms are defined as

\[ f_{y_1, y_2}(y_1, y_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(F, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2) \, dt_1 dt_2 \quad (A4) \]

and

\[ a_{y_1, y_2}(y_1, y_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(A, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2) \, dt_1 dt_2. \quad (A5) \]

We take derivatives with respect to \( y_1 \) and \( y_2 \) and obtain

\[ (-1)^{r+s} \frac{d^{r+s} a(y_1, y_2)}{dy_1^r dy_2^s} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(A, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2) (it_1)^r (it_2)^s \, dt_1 dt_2 \quad (A6) \]

Multiplying (A3) by \( \exp(-it_1 y_1 - it_2 y_2) \), integrating the resulting expression, and applying (A4), (A5), and (A6) yield

\[ f_{y_1, y_2}(y_1, y_2) = a_{y_1, y_2}(y_1, y_2) + \sum_{r+s > 0} E_{rs} \frac{(-1)^{r+s}}{r! s!} \frac{d^{r+s} a(y_1, y_2)}{dy_1^r dy_2^s}. \]

The above amounts to the inversions of the Fourier transforms involved.  

**Appendix B.** The relations between the correction factors and the product

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The relations for $E_{01}$, $E_{02}$, $E_{03}$, $E_{04}$, $E_{12}$, and $E_{13}$ can be found by symmetry and hence omitted.

**Proof of Lemma 1.** Truncating the summation term of (3) at $r + s = 4$ yields (4). To derive the rest of the results, we note that the exponential
function in (A3) is analytic and hence expandable. Thus we have

\[
\exp \left( \prod_{r+s \geq 0} \left( \kappa_{r,s}(F) - \kappa_{r,s}(A) \right) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \prod_{r+s \geq 0} \left( \kappa_{r,s}(F) - \kappa_{r,s}(A) \right) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \right) \frac{1}{n!}
\]

\[
= \sum_{r+s \geq 0} E_{rs} \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \frac{1}{n!}
\]

By matching the coefficients for the term \( t_1^r t_2^s \), we obtain the relations for the case when \( 0 < r + s \leq 4 \).

\[ \square \]

**Appendix C. Proof of Theorem 2.** As \( e^{y_2} - e^{y_1} - K \geq 0 \) implies \( y_2 \geq \ln (e^{y_1} - K) \), it follows that

\[
G_{Y_1}(y_1) = \int_{\ln(e^{y_1} - K)}^{\infty} (e^{y_2} - e^{y_1} - K) a(y_2|y_1)dy_2
\]

\[
= \int_{\ln(e^{y_1} - K)}^{\infty} e^{y_2} a(y_2|y_1)dy_2 - (e^{y_1} + K) \int_{\ln(e^{y_1} - K)}^{\infty} a(y_2|y_1)dy_2
\]

Using (10), we do a complete-the-square, and a change of variable. This will produce the identity

\[
\int_{\ln(e^{y_1} - K)}^{\infty} e^{y_2} a(y_2|y_1)dy_2 = e^{A_2(y_1)} N(x_1(y_1)).
\]

The identity

\[
\int_{\ln(e^{y_1} - K)}^{\infty} a(y_2|y_1)dy_2 = N(x_2(y_1))
\]

follows from a similar change-of-variable.

\[ \square \]

**Appendix D.** The following are results needed for computation of adjustment for skewness shown in Section 4.2. The \( q' \)'s are given in closed form.
The derivations of the above can be found in Xie (2008).

Appendix E. The following are results needed for computation of adjustment for kurtosis shown in Section 4.3. The $u_i'$s are given in closed form.
\[
\begin{align*}
\mathbf{u}_1 &= \frac{1}{(1-\rho^2)^2} \left[ \frac{E_{40}}{8\sigma_1^4} + \frac{E_{04}}{8\sigma_1^4} + \frac{E_{22} (1 + 2\rho^2)}{4\sigma_1^2 \sigma_2^2} - \frac{E_{31}\rho}{2\sigma_1^2 \sigma_2} - \frac{E_{13}\rho}{2\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_2 &= \frac{1}{(1-\rho^2)^3 \sigma_1^2} \left[ \frac{-E_{40}}{4\sigma_1^4} - \frac{E_{04}\rho^2}{4\sigma_1^4} + \frac{E_{22} (1 + 5\rho^2)}{4\sigma_1^2 \sigma_2^2} + \frac{E_{31}\rho}{2\sigma_1^2 \sigma_2} + \frac{E_{13}\rho(1 + \rho^2)}{2\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_3 &= \frac{\rho}{(1-\rho^2)^3 \sigma_1 \sigma_2} \left[ \frac{E_{40}}{2\sigma_1^4} + \frac{E_{04}}{2\sigma_1^4} + \frac{E_{22} (2 + \rho^2)}{\sigma_1^2 \sigma_2^2} - \frac{E_{31}(1 + 3\rho^2)}{2\sigma_1^2 \sigma_2} - \frac{E_{13}(1 + 3\rho^2)}{2\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_4 &= \frac{1}{(1-\rho^2)^3 \sigma_2^2} \left[ \frac{-E_{40}\sigma^2}{4\sigma_1^4} - \frac{E_{04}\rho^2}{4\sigma_1^4} - \frac{E_{22} (1 + 5\rho^2)}{4\sigma_1^2 \sigma_2^2} + \frac{E_{31}(1 + \rho^2)}{2\sigma_1^2 \sigma_2} + \frac{E_{13}\rho}{2\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_5 &= \frac{1}{(1-\rho^2)^4 \sigma_1^2 \sigma_2^2} \left[ \frac{E_{40}\sigma^2}{4\sigma_1^4} + \frac{E_{04}\rho^2}{4\sigma_1^4} + \frac{E_{22} (1 + 4\rho^2 + \rho^4)}{4\sigma_1^2 \sigma_2^2} - \frac{E_{31}\rho(1 + \rho^2)}{2\sigma_1^2 \sigma_2} - \frac{E_{13}\rho(1 + \rho^2)}{2\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_6 &= \frac{\rho}{2(1-\rho^2)^4 \sigma_1 \sigma_2} \left[ \frac{-E_{40}}{3\sigma_1^6} - \frac{E_{04}\rho^2}{3\sigma_1^6} - \frac{E_{22}(1 + \rho^2)}{\sigma_1^4 \sigma_2^2} + \frac{E_{31}(1 + 3\rho^2)}{3\sigma_1^4 \sigma_2^2} + \frac{E_{13}\rho(3 + \rho^2)}{3\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_7 &= \frac{\rho}{2(1-\rho^2)^4 \sigma_1 \sigma_2} \left[ \frac{-E_{40}\rho^2}{3\sigma_1^6} - \frac{E_{04}}{3\sigma_1^6} - \frac{E_{22}(1 + \rho^2)}{\sigma_1^4 \sigma_2^2} + \frac{E_{31}(1 + 3\rho^2)}{3\sigma_1 \sigma_2^3} + \frac{E_{13}\rho(3 + \rho^2)}{3\sigma_1 \sigma_2^3} \right] \\
\mathbf{u}_8 &= \frac{1}{(1-\rho^2)^4 \sigma_2^2} \left[ \frac{E_{40}\rho^4}{24\sigma_1^6} + \frac{E_{04}}{24\sigma_1^6} + \frac{E_{22}\rho^2}{6\sigma_1^4 \sigma_2^4} - \frac{E_{31}\rho^3}{6\sigma_1^4 \sigma_2^2} - \frac{E_{13}\rho}{6\sigma_1^4 \sigma_2^2} \right] \\
\mathbf{u}_9 &= \frac{1}{(1-\rho^2)^4 \sigma_1^2} \left[ \frac{E_{40}\rho^4}{24\sigma_1^6} + \frac{E_{04}\rho^4}{24\sigma_1^6} + \frac{E_{22}\rho^2}{6\sigma_1^4 \sigma_2^4} - \frac{E_{31}\rho^3}{6\sigma_1 \sigma_2^3} - \frac{E_{13}\rho}{6\sigma_1 \sigma_2^3} \right].
\end{align*}
\]

and \(J_{Y_2}(y_1)\) is defined by

\[
J_{Y_2}(y_1) = \int_{\infty}^{-\infty} (e^{y_2} - e^{y_1} - K)^+ (y_2 - \mu_2)^2 a_{Y_2|Y_1}(y_2|y_1)dy_2
\]

The derivations of the above can be found in Xie (2008).

**Appendix F.** Derivations for results shown in Section 6.2. The
following are the derivations of higher moments of \((Y_1, Y_2)\) under \(F\):

\[
\begin{align*}
\mu_{21}(F) &= E \left[ (\Omega_1 Z_1 + L_1 - m_{11})^2 (\Omega_2 Z_2 + L_2 - m_{21}) \right] \\
&= E \left[ \Omega_1^2 \Omega_2 Z_1^2 Z_2 + \Omega_1^2 Z_1^2 (L_2 - m_{21}) + 2 \Omega_1 \Omega_2 Z_1 Z_2 (L_1 - m_{11}) \right] \\
&\quad + 2 \Omega_1 Z_1 (L_1 - m_{11})(L_2 - m_{21}) + (L_1 - m_{11})^2 \Omega_2 Z_2 + (L_1 - m_{11})^2 (L_2 - m_{21}) \\
&= 0 \\
\mu_{30}(F) &= E \left[ (\Omega_1 Z_1 + L_1 - m_{11})^3 \right] \\
&= E \left[ \Omega_1^3 Z_1^3 \right] + E \left[ (L_1 - m_{11})^3 \right] + E \left[ 3(\Omega_1 Z_1)^2 (L_1 - m_{11}) \right] + E \left[ 3 \Omega_1 Z_1 (L_1 - m_{11})^2 \right] \\
&= E \left[ (L_1 - m_{11})^3 \right] \\
&= E[L_1^3] + 3m_{11}^2 E[L_1] - 3m_{11} E[L_1^2] - m_{11}^3 \\
&= m_{13} + 3m_{11}^2 - 3m_{11} m_{12} - m_{11}^3 \\
&= m_{13} + 2m_{11} - 3m_{11} m_{12} \\
&= 6\lambda T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) + 6 (\lambda T)^2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \left( \frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2} \right) + (\lambda T)^3 \left( \frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3} \right), \\
&\quad + 2 (\lambda T)^3 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^3 \\
&\quad - 3 (\lambda T) \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \left[ 2 \lambda T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + (\lambda T)^2 \left( \frac{p_1}{\eta_{11}^2} - \frac{q_1}{\eta_{12}^2} \right) \right] \\
&= 6\lambda T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)
\end{align*}
\]
\[ \mu_{40}(F) = E[(\Omega_1 Z_1 + L_1 - m_{11})^4] \]
\[
= E[\Omega_1^2 Z_1^2] + E[(L_1 - m_{11})^4] + 4E[(\Omega_1 Z_1)^3 (L_1 - m_{11})] \]
\[
+ 4E[\Omega_1 Z_1 (L_1 - m_{11})^3] + 6E[\Omega_1^2 Z_1^2 (L_1 - m_{11})^2] \]
\[
= 3\Omega_1^4 + E[(L_1 - m_{11})^4] + 6\Omega_1^2 E[(L_1 - m_{11})^2] \]
\[
= 3\Omega_1^4 + E[L_1^4] + 6m_{11}^2 E[L_1^2] - 4m_{11}E[L_1^3] - 4m_{11}^3 E[L_1] + 6\Omega_1^2 (E[L_1^2] - m_{11}^2) \]
\[
= 3\Omega_1^4 + m_{14} + 6m_{11}^2 m_{12} - 4m_{11} m_{13} - 4m_{11}^4 + 6\Omega_1^2 (m_{12} - m_{11}) \]
\[
= 3\Omega_1^4 + 24\lambda_1 T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + 12(\lambda_1 T)^2 \left[ 2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) + \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right)^2 \right] \]
\[
+ 12(\lambda_1 T)^3 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + (\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4 \]
\[
+ 6(\lambda_1 T)^2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^2 \left[ 2\lambda_1 T \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + (\lambda_1 T)^2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^2 \right] - 4\lambda_1 T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \]
\[
\times \left[ 6\lambda_1 T \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) + 6(\lambda_1 T)^2 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + (\lambda_1 T)^3 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^3 \right] \]
\[
- 4(\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4 \]
\[
+ 6\Omega_1^2 (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) \]
\]

After cancellations, we find
\[
\mu_{40}(F) = 3\Omega_1^4 + 12\lambda_1 T \left[ 2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + \Omega_1^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) \right] \]
\[
+ 12(\lambda_1 T)^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right)^2 - (\lambda_1 T)^4 \left( \frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}} \right)^4 \]
\[ \mu_{22}(F) = E [(\Omega_1 Z_1 + L_1 - m_{11})^2 (\Omega_2 Z_2 + L_2 - m_{21})^2] \]
\[ = \Omega_1^2 \Omega_2^2 E [Z_1^2 Z_2^2] + \Omega_1^2 E [Z_1^2 (L_2 - m_{21})^2] + \Omega_2^2 E [Z_2^2 (L_1 - m_{11})^2] + E [(L_1 - m_{11})^2 (L_2 - m_{21})^2] \]
\[ = (1 + 2\mu^2) \Omega_1^2 \Omega_2^2 + \Omega_1^2 E [(L_2 - m_{21})^2] + \Omega_2^2 E [(L_1 - m_{11})^2] + E [(L_1 - m_{11})^2 (L_2 - m_{21})^2] \]
\[ = (1 + 2\mu^2) (\Omega_1 \Omega_2)^2 + \Omega_1^2 (2\lambda_2 T) \left( \frac{p_2}{\eta_{21}} + \frac{q_2}{\eta_{22}} \right) + \Omega_2^2 (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) + 4\lambda_1 \lambda_2 T^2 \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right) \left( \frac{p_2}{\eta_{21}} + \frac{q_2}{\eta_{22}} \right) \]

and finally, we have

\[ \mu_{31}(F) = E [(\Omega_1 Z_1 + L_1 - m_{11})^3 (\Omega_2 Z_2 + L_2 - m_{21})] \]
\[ = \Omega_1^3 \Omega_2 E [Z_1^3 Z_2] + 3\Omega_1 \Omega_2 E [Z_1 (L_1 - m_{11})^2 Z_2] \]
\[ = 3\rho \Omega_1^3 \Omega_2 + 3\rho \Omega_1 \Omega_2 (2\lambda_1 T) \left( \frac{p_1}{\eta_{11}} + \frac{q_1}{\eta_{12}} \right). \]