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Lévy-driven non-Gaussian Ornstein–Uhlenbeck processes for degradation-based reliability analysis

Yin Shu[a], Qianmei Feng[a], Edward P.C. Kao[b] and Hao Liu[c]

[a]Department of Industrial Engineering, University of Houston, Houston, TX, USA; [b]Department of Mathematics, University of Houston, Houston, TX, USA; [c]Division of Biostatistics, Dan L. Duncan Cancer Center, Baylor College of Medicine, Houston, TX, USA

ABSTRACT

We use Lévy subordinators and non-Gaussian Ornstein–Uhlenbeck processes to model the evolution of degradation with random jumps. The superiority of our models stems from the flexibility of such processes in the modeling of stylized features of degradation data series such as jumps, linearity/nonlinearity, symmetry/asymmetry, and light/heavy tails. Based on corresponding Fokker–Planck equations, we derive explicit results for the reliability function and lifetime moments in terms of Laplace transforms, represented by Lévy measures. Numerical experiments are used to demonstrate that our general models perform well and are applicable for analyzing a large number of degradation phenomena. More important, they provide us with a new methodology to deal with multi-degradation processes under dynamic environments.

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Lévy subordinators; non-Gaussian OU processes; Fokker–Planck equations; Laplace transform; degradation processes; lifetime moments

1. Introduction

Stochastic processes have been extensively applied to model the temporal variability of degradation or deterioration evolution of engineering structures and infrastructures (Esary *et al.*, 1973; Abdel-Hameed, 1975; Cholette and Djurdjanovic, 2014; Liu *et al.*, 2014; Moghaddass and Zuo, 2014; Giorgio *et al.*, 2015). The special cases of Lévy processes such as the Wiener process/Brownian motion with drift, the compound Poisson process, and the gamma process have been extensively studied. These special stochastic models are limited to represent degradation without jumps (the Wiener process) and degradation with Poisson-type (the compound Poisson process) or gamma-type (the gamma process) jumps. They have independent and stationary increments, which makes them suitable to model degradation processes with linear mean paths. To overcome the limitation from the linear mean property, Gaussian Ornstein–Uhlenbeck (OU) processes driven by a Wiener process have been developed for survival analysis (Aalen and Gjessing, 2004). However, the assumptions of no jumps and Gaussian distribution (symmetric and light-tailed; i.e., all of the positive moments are finite) are not consistent with many degradation phenomena. In this article, in order to flexibly handle stylized features of degradation data series such as complex jumps, linearity/nonlinearity, symmetry/asymmetry, and light/heavy tails, we propose to model stochastic degradation with independent or dependent increments using Lévy subordinators or OU processes driven by Lévy subordinators (i.e., non-Gaussian OU processes), respectively. For these general stochastic degradation processes, we construct systematic procedures to derive the explicit expressions for reliability function and lifetime moments using Fokker–Planck equations. Our proposed new models offer a general approach for modeling stochastic degradation with complex jump mechanisms using a broad class of Lévy processes and their functional extensions.

In reliability studies, a Wiener process has been used to model degradation without jumps that changes non-monotonically according to Gaussian laws (Whitmore *et al.*, 1998; Jin and Matthews, 2014). Some other degradation models without considering jumps were studied (Chen and Tsui, 2013; Wang *et al.*, 2016; Zeng *et al.*, 2016). A compound Poisson process has been applied to model a finite number of jumps that occur based on Poisson laws (Esary *et al.*, 1973; Sobczyk, 1987). A gamma process has been widely used for modeling degradation processes that progress in one direction with an infinite number of jumps in any finite time intervals (Van Noortwijk, 2009; Ye *et al.*, 2012; Ye *et al.*, 2014). Recently, Kharoufeh (2003), Kharoufeh *et al.* (2006), Kharoufeh and Mixon (2009), and Kharoufeh *et al.* (2013) obtained explicit results for both life distribution and lifetime moments, assuming a linear degradation with Poisson-type jumps. In practice, however, many different complex jump mechanisms are embedded in continuous degradation processes, beyond Poisson and gamma types. Existing stochastic degradation models are not appropriate to model such situations.

Lévy processes provide a potential candidate to describe a broad class of degradation with random jumps. The theories of Lévy processes were developed in Sato (1999) and Applebaum (2009), and they have been widely applied in the fields of economics and finance (Schoutens, 2003; Cont and Tankov, 2004). Çinlar (1977) was the first study to use Lévy processes in degradation analysis. Abdel-Hameed (1984) studied the life distribution properties of devices subject to Lévy degradation. However, Lévy processes have not been well-developed

CONTACT Qianmei Feng qfeng@central.uh.edu
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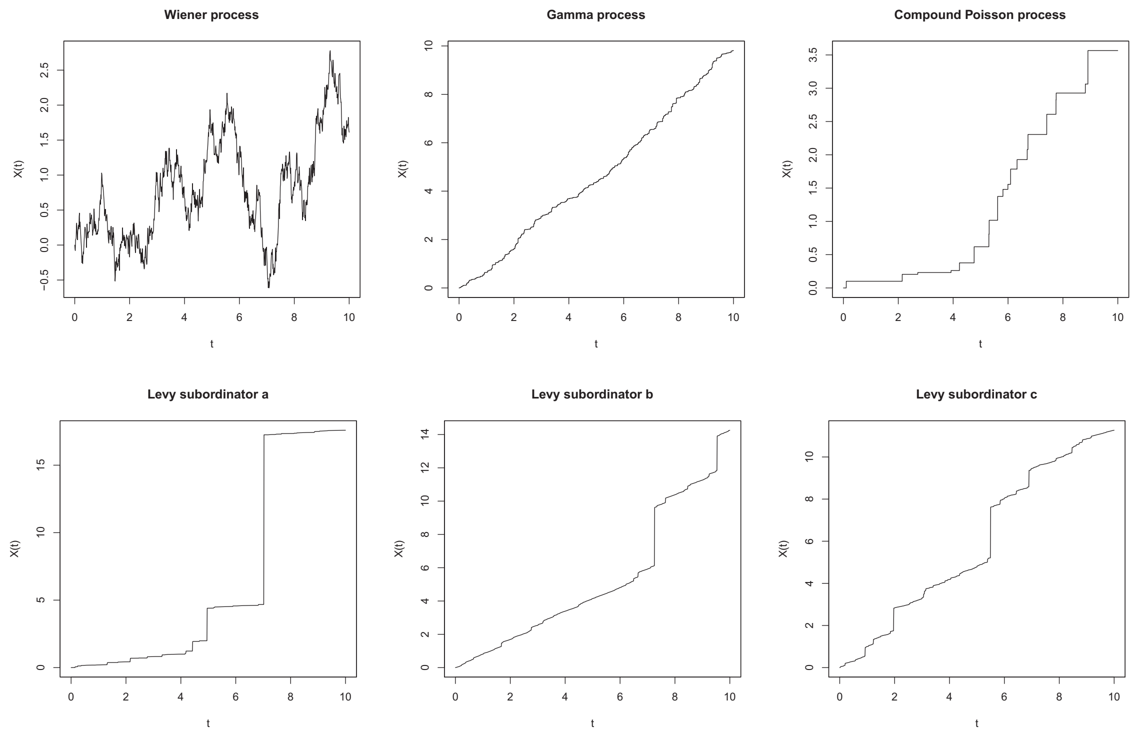

Figure 1. Sample paths of Lévy processes (Wiener process (0, 1); gamma process (20, 20); compound Poisson process with jumps density 2 and jumps size following gamma distribution (1, 10); Lévy subordinator a: inverse Gaussian process (0.5, 0.1); Lévy subordinator b: positive stable process (0.9); Lévy subordinator c: positive stable process (0.92)).

for use in degradation modeling; e.g., no explicit results of life distribution from general Lévy degradation processes. Shu *et al.* (2015) gave a new closed-form reliability function for degradation described by Lévy subordinators, a class of nondecreasing Lévy processes, which was consistent with the observed physical degradation phenomena. The advantages of using Lévy subordinators were also demonstrated. With independent and stationary increments, however, all Lévy processes have linear mean paths; i.e., the mean of a Lévy processes is linear with respect to time t. To model degradation processes with nonlinear mean paths in general, a functional extension of Lévy subordinators, non-Gaussian OU processes, is an interesting and effective model to address the problem.

OU processes, another important class of continuous-time continuous-state stochastic processes, named after Ornstein and Uhlenbeck (1930), are used in a physical modeling context, where the background driving process is a Wiener process and thus is called an ordinary or Gaussian OU process (Maller *et al.*, 2009). In the field of physics, the ordinary OU process is represented by the classic Klein–Kramers dynamics (Kramers, 1940). In the field of finance, the process is known as the Vasicek model (Vasicek, 1977), with the interest rate being modeled by such a process. Non-Gaussian OU processes are a generalization of ordinary OU processes that are obtained by replacing Wiener processes with non-Gaussian Lévy processes (i.e., Lévy processes without a Gaussian part; e.g., positive tempered stable processes). They have been recently developed and applied in financial models, by Barndorff-Nielsen and Shephard (2001, 2002, 2003). To the best of our knowledge, non-Gaussian OU processes have not been used in degradation modeling. In fact, it is nontrivial to obtain a closed-form distribution function for an OU process driven by a Lévy process.

Fokker–Planck equations provide us with a way to analyze probability laws for stochastic processes, especially for those without closed-form distributions. Fokker-Planck equations are a fascinating topic that is being studied by mathematicians in the field of stochastic processes. As Partial Differential Equations (PDEs) of the probability density functions, they describe the time evolution of probability density for stochastic processes and are thus useful in quantifying random phenomena, such as the propagation of uncertainty. The Fokker–Planck equations for Weiner-based processes can be found in many textbooks (Risken, 1996; Klebaner, 2005). For such processes, it is straightforward to derive the Fokker–Planck equations, due to the absence of jump mechanisms. However, for Lévy-based processes, explicit results of Fokker–Planck equations cannot be easily derived, due to the difficulty in obtaining the expression for the adjoint operators of the infinitesimal generators associated with Lévy-based processes (Sun and Duan, 2012). Some interesting results on Fokker–Planck equations for Lévy-based processes can be found in Schertzer *et al.* (2001), Denisov *et al.* (2009), Sun and Duan (2012). Ren *et al.* (2012) gave a numerical algorithm to calculate the mean exit time for Lévy systems.

In this article, we consider a single degradation process with random jumps in a system; i.e., a process of stochastically continuous degradation with sporadic jumps that occur at random times and have random sizes. The system fails when the degradation process hits a boundary. We first use Lévy subordinators, a class of Lévy processes with nondecreasing sample paths, to model the evolution of the degradation with linear mean paths (Fig. 1). We then propose a functional extension of Lévy subordinators, non-Gaussian OU processes (OU processes driven by Lévy subordinators),

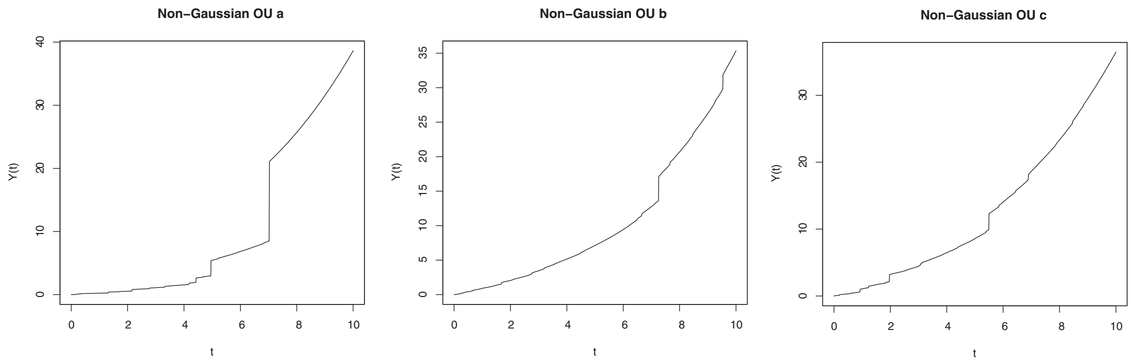

Figure 2. Sample paths of non-Gaussian OU processes (non-Gaussian OU process a: OU process driven by inverse Gaussian process (0.5, 0.1) and $\alpha = 0.2$; non-Gaussian OU process b: OU process driven by positive stable process (0.9) and $\alpha = 0.1$; non-Gaussian OU process c: OU process driven by positive stable process (0.92) and $\alpha = 0.1$).

to model degradation processes with nonlinear mean paths (Fig. 2).

Figure 1 shows sample paths of three commonly used Lévy processes (Wiener process, gamma process, and compound Poisson process) and three Lévy subordinators with different jump mechanisms specified by different Lévy measures. Figure 2 illustrates sample paths of OU processes driven by Lévy subordinators, a class of non-Gaussian OU processes, and they are the solutions of stochastic differential equations (SDEs) driven by Lévy subordinators. The sample data are simulated using R(YUIMA) (Brouste *et al.*, 2014). In practice, many degradation processes in highly reliable systems have similar paths to those shown in Fig. 2: they increase slowly at the early stage but increase sharply when degradation is accumulated. In these cases, the linear mean path of a Lévy subordinator is not appropriate to represent the degradation.

For both general Lévy subordinators and non-Gaussian OU processes, the probability distributions are not analytically available. In addition, the analytical derivation is intractable for non-Gaussian OU processes. In this article, we tackle these challenges by using the corresponding Fokker–Planck equations and then derive explicit expressions for reliability function and lifetime moments in terms of the Laplace transform. The results are compact enough to be able to easily compute and evaluate reliability characteristics. More important, by introducing Fokker–Planck equations into stochastic degradation analysis, our work provides a new methodology for reliability analysis of complex degradation phenomenon, such as multi-degradation processes under dynamic environments.

The organization of the rest of this article is as follows. Section 2 begins with the Lévy–Itô decomposition and then describes model construction. In Section 3, we derive the explicit expressions of the reliability function and lifetime moments for systems subject to degradation described by Lévy subordinators and non-Gaussian OU processes, respectively, based on Fokker–Planck equations. Numerical examples are illustrated in Section 4, and conclusions are given in Section 5.

Notations

- Euclidean space: R^d, $d \in N$;
- Euclidean norm: $|x| = (x, x)^{1/2} = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$;
- indicator function: $I_A(x)$;
- Lévy processes: $X(t)$;
- Lévy subordinators: $X_s(t)$;
- Lévy measure: ν;
- non-Gaussian OU processes (OU processes driven by Lévy subordinators): $Y(t)$.

2. Preliminaries

2.1. Lévy–Itô decomposition

We begin with the definition of Poisson random measure from Sato (1999). A random variable J has a Poisson distribution with a mean 0 if $J = 0$ almost surely (a.s.) and J has a Poisson distribution with a mean $+\infty$ if $J = +\infty$ a.s.

Definition 1 (Sato, 1999). Let $(\Theta, \mathbb{B}, \nu)$ be a σ-finite measure space. Given $\bar{Z}_+ = \{0, 1, 2, \ldots, +\infty\}$, a family of $\bar{Z}_+$-valued random variables $\{J(A) : A \in \mathbb{B}\}$ is called a Poisson random measure on Θ with an intensity measure ν, if the following conditions hold:

- for every A, $J(A)$ has a Poisson distribution with a mean $\nu(A)$;
- if $A_1, A_2, \ldots A_n$ are disjoint, then $J(A_1), J(A_2), \ldots J(A_n)$ are independent; and
- for every ω, $J(\cdot, \omega)$ is a measure on Θ.

Lemma 1. (The Lévy–Itô decomposition (Applebaum, 2009)). *If $X(t)$ is a Lévy process, then there exist $b \in R^d$, a Brownian motion B_a with a covariance matrix a, and an independent Poisson random measure J on $R^+ \times R^d$ such that, for each $t \geq 0$,*

$$X(t) = bt + B_a(t) + \int_{|y|<1} y(J(t, dy) - \nu(t, dy)) + \int_{|y|\geq 1} yJ(t, dy),$$

where $\nu(t, dy)$ is the mean of the Poisson random measure $J(t, dy)$.

The intensity measure $\nu(t, dy)$ is often called the Lévy measure. Based on the property of independent and stationary increments of Lévy process and from the Lévy–Khintchine formula (Sato, 1999), $\nu(t, dy) = \nu(dy)t$.

2.2. Model construction

We assume that there is a single degradation path with random jumps occurring in a system. We use a Lévy subordinator $X_s(t)$ and a non-Gaussian OU process $Y(t)$ to model the degradation evolution with linear and nonlinear mean paths, respectively.

Lemma 2 (Sato, 1999). *Let $d = 1$. A Lévy process is a subordinator if and only if $a = 0$, $\int_{(-\infty,0)} \nu(dy) = 0$, $\int_{R^+} \min\{1, y\}\nu(dy) < \infty$, and the drift $b - \int_{0<y<1} y\nu(dy) \geq 0$.*

Based on Lemmas 1 and 2, for Lévy subordinator $X_s(t)$:

$$X_s(t) = bt + \int_{0<y<1} y(J(t, dy) - \nu(t, dy)) + \int_{y\geq 1} yJ(t, dy). \tag{1}$$

In Equation (1), the continuous degradation is modeled by $(b - \int_{0<y<1} y\nu(dy))t$, and the random jumps are modeled by the Poisson random measure $\int_{R^+} yJ(t, dy)$.

If we specify $\nu(dx) = \gamma x^{-1}e^{-\beta x}dx$ for small jumps in an infinitesimal time interval, then the Lévy subordinator in Equation (1) is a temporally homogeneous gamma process (a gamma process with stationary increments) $G(t)$, which has a density

$$f_{G(t)} = Ga\left(x|\gamma t, \beta\right) = \frac{\beta^{\gamma t} x^{\gamma t-1} e^{-\beta x}}{\Gamma\left(\gamma t\right)}, \gamma, \beta, x, t > 0.$$

For big jumps occurring based on the Poisson law, we can specify $\nu(dx) = \lambda\mu(dx)$, and then the Lévy subordinator is a compound Poisson process $C(t)$, which has a jump density λ and a jump size distribution μ. Moreover, we can specify different forms of Lévy measures in order to model different complex jump mechanisms.

A non-Gaussian OU process $Y(t)$ is the solution of an SDE driven by $X_s(t)$:

$$dY(t) = \alpha Y(t)dt + dX_s(t). \tag{2}$$

Proposition 1. *The non-Gaussian OU process resulted from Equation (2) is*

$$Y(t) = e^{\alpha t}Y(0) + \int_0^t e^{\alpha(t-\xi)} dX_s(\xi).$$

Proof. If $f(t, y) \in C^{1,2}$, then based on Taylor series:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}dy + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dy)^2 + \frac{\partial^2 f}{\partial y \partial t}dydt.$$

Let $f(t, y) = ye^{-\alpha t}$, then

$$\frac{\partial f}{\partial t} = -\alpha y e^{-\alpha t}, \quad \frac{\partial f}{\partial y} = e^{-\alpha t}, \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

Then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial y}dy = -\alpha y e^{-\alpha t}dt + e^{-\alpha t}dy = e^{-\alpha t}dx,$$

$$y_t = e^{\alpha t}y_0 + e^{\alpha t}\int_0^t e^{-\alpha\xi}dx_\xi.$$

This completes the proof. □

$Y(0)$ represents the initial state of the degradation, and we assume $Y(0) = 0$ a.s. as many new systems have not accumulated degradation when they are first used. We assume $\alpha > 0$, which guarantees that the degradation process is nondecreasing:

$$\begin{aligned} Y(t) &= \int_0^t e^{\alpha(t-\xi)} dX_s(\xi) \\ &= \int_0^t e^{\alpha(t-\xi)} \left(bd\xi + \int_{0<y<1} y\left(J(d\xi, dy) - \nu(d\xi, dy)\right) + \int_{y\geq 1} yJ\left(d\xi, dy\right) \right) \\ &= \frac{1}{\alpha}(e^{\alpha t} - 1)\left(b - \int_{0<y<1} y\nu(dy)\right) + \int_0^t e^{\alpha(t-\xi)} \int_{R^+} yJ\left(d\xi, dy\right). \end{aligned} \tag{3}$$

In Equation (3), the continuous degradation part is modeled by

$$\frac{b - \int_{0<y<1} y\nu(dy)}{\alpha}(e^{\alpha t} - 1),$$

and the random jumps are modeled by the Poisson random measure $\int_0^t e^{\alpha(t-\xi)} \int_{R^+} yJ(d\xi, dy)$. As illustrated in Fig. 2, the mean degradation path of $Y(t)$ is exponential with respect to (w.r.t.) t, instead of linear of $X_s(t)$.

3. Reliability function and lifetime moments

The system fails when the degradation process $X_s(t)$ or $Y(t)$ exceeds a failure threshold x or y. To simplify formulas, we assume that the failure threshold is a constant, and it is straightforward to extend our models when the failure threshold is a random variable. Based on $X_s(t)$, the lifetime of the system and its moments are defined respectively as

$$T_x = \inf\left\{t : X_s(t) > x\right\}, \ M\left(T_X^n, x\right) = E\left(T_x^n\right). \tag{4}$$

Since $X_s(t)$ is nondecreasing, we have $\{T_x \geq t\} \equiv \{X_s(t) \leq x\}$. Then the reliability function can be defined as

$$R_X(x, t) = P(T_x \geq t) = P(X_s(t) \leq x) = F_{X_s(t)}(x). \tag{5}$$

Based on $Y(t)$, similar definitions are

$$T_y = \inf\{t : Y(t) > y\}, \ M\left(T_Y^n, y\right) = E\left(T_y^n\right), \tag{6}$$

$$R_Y(y, t) = P(T_y \geq t) = P(Y(t) \leq y) = F_{Y(t)}(y). \tag{7}$$

For many new systems that have not accumulated degradation when they are first operated, we have $X_s(0) = 0$ a.s., and

$$R_X(x, 0) = P(X_s(0) \leq x) = F_{X_s(0)}(x) = h(x),$$

$$p(x, 0) = \frac{\partial F_{X_s(0)}(x)}{\partial x} = \delta(x),$$

where $h(x) = I_{[0,\infty)}(x)$ is the unit step function (or the Heaviside step function), and $\delta(x)$ is the Dirac delta function. Similarly, we have

$$R_Y(y, 0) = P(Y(0) \leq y) = F_{Y(0)}(y) = h(y),$$

$$p(y, 0) = \frac{\partial F_{Y(0)}(y)}{\partial y} = \delta(y).$$

In addition, $R_X(0,t) = P(T_0 \geq t) = P(X_s(t) \leq 0) = I_{(-\infty,0]}(t)$, and $R_Y(0,t) = P(T_0 \geq t) = P(Y(t) \leq 0) = I_{(-\infty,0]}(t)$.

To obtain expressions of reliability functions and lifetime moments in Equations (4), (5), (6), and (7), we need to study the probability laws of $X_s(t)$ and $Y(t)$. Since there are no closed-form distribution functions for general Lévy subordinators, it is a challenge to derive the explicit expressions for reliability functions and lifetime moments. As PDEs of probability density functions, Fokker–Planck equations (Sun and Duan, 2012) provide us a way to overcome the challenge in analyzing probability laws for stochastic processes we are interested in, especially for those without closed-form distributions. The Fokker–Planck equation, also known as the Kolmogorov forward equation, describes the time evolution of probability density for stochastic processes.

Let L be an operator and L^* be the adjoint operator of L, then

$$\int_R Lf(x)g(x)dx = \int_R f(x)L^*g(x)dx.$$

Let $p(x,t)$ be the probability density function for a stochastic process $X(t)$, and the Fokker–Planck equation is

$$\frac{\partial p(x,t)}{\partial t} = L^* p(x,t),$$

where L^* is the adjoint operator of the infinitesimal generator L of $X(t)$:

$$Lf(x) = \lim_{\Delta t \to 0} \frac{E\left\{f\left(X_{t+\Delta t}\right)|X_t = x\right\} - f(x)}{\Delta t}.$$

The Laplace transform of $p(x,t)$ w.r.t. t is defined to be

$$p^L(x,\omega) = \int_{R^+} e^{-\omega t} p(x,t)dt, \quad \omega > 0.$$

The Laplace transform of $p^L(x,\omega)$ w.r.t. x is

$$p^{LL}(u,\omega) = \int_{R^+} e^{-ux} p^L(x,\omega)dx, \quad u > 0.$$

Lemma 3. *Let $R^{LL}(u,\omega)$ be the Laplace expression of reliability function $R(x,t)$, then*

$$R^{LL}(u,\omega) = u^{-1}p^{LL}(u,\omega).$$

Proof. From the definition of the reliability function, we have:

$$p(x,t) = \frac{\partial R(x,t)}{\partial x}.$$

Then

$$p^L(x,\omega) = \int_{R^+} e^{-\omega t} p(x,t)dt = \int_{R^+} e^{-\omega t} \frac{\partial R(x,t)}{\partial x} dt$$
$$= \frac{\partial R^L(x,\omega)}{\partial x}.$$

We have

$$p^{LL}(u,\omega) = \int_{R^+} e^{-ux} \frac{\partial R^L(x,\omega)}{\partial x} dx = \int_{R^+} e^{-ux} dR^L(x,\omega)$$
$$= e^{-ux} R^L(x,\omega)|_{R^+} - \int_{R^+} R^L(x,\omega) de^{-ux}$$
$$= uR^{LL}(u,\omega).$$

□

3.1. Results based on Lévy subordinators

For degradation with random jumps described by a Lévy subordinator $X_s(t)$, we derive the explicit expressions of $R_X(x,t)$ and lifetime moments $M(T_X^n, x)$ in terms of Laplace transform, represented by Lévy measures. Using the procedure similar to Kharoufeh (2003), the results are presented in Theorems 1 and 2.

Theorem 1. *For degradation with random jumps described by a Lévy subordinator, the Laplace expression of reliability function is*

$$R_X^{LL}(u,\omega) = u^{-1}\left\{\omega + b^*u - \int_{R^+} (e^{-uy} - 1)\nu(dy)\right\}^{-1},$$

where $b^* \geq 0$, ν is the Lévy measure.

Proof. Let $p(x,t)$ be the probability density function of a Lévy subordinator $X_s(t)$. Based on Sun and Duan (2012), the Fokker–Planck equation for $X_s(t)$ is

$$\frac{\partial p(x,t)}{\partial t} = -b\frac{\partial p(x,t)}{\partial x} + \int_{R^+} \left[p(x-y,t) - p(x,t) + I_{y\in(0,1)} y \frac{\partial p(x,t)}{\partial x} \right] \nu(dy). \tag{8}$$

For Equation (8), we perform a Laplace transform of $p(x,t)$ w.r.t. t for both sides:

$$\omega p^L(x,\omega) - p(x,0) = -b\frac{\partial p^L(x,\omega)}{\partial x} + \int_{R^+} \left[p^L(x-y,\omega) - p^L(x,\omega) + I_{y\in(0,1)} y \frac{\partial p^L(x,\omega)}{\partial x} \right] \nu(dy). \tag{9}$$

For Equation (9), we perform a Laplace transform of $p^L(x,\omega)$ w.r.t. x for both sides; then

$$\omega p^{LL}(u,\omega) - 1 = -bup^{LL}(u,\omega) + \int_{R^+} \left[e^{-uy} p^{LL}(u,\omega) - p^{LL}(u,\omega) + I_{y\in(0,1)} yup^{LL}(u,\omega) \right] \nu(dy).$$

Let $b^* = b - \int_{0<y<1} y\nu(dy)$, then

$$p^{LL}(u,\omega) = \left\{\omega + b^*u - \int_{R^+} (e^{-uy} - 1)\nu(dy)\right\}^{-1}.$$

Based on Lemma 3, we obtain

$$R_X^{LL}(u,\omega) = u^{-1}\left\{\omega + b^*u - \int_{R^+} (e^{-uy} - 1)\nu(dy)\right\}^{-1}.$$

□

Remark 1. For Equation (8), we perform a Laplace transform of $p(x,t)$ w.r.t. x for both sides:

$$E[e^{-uX(t)}] = p^L(u,t) = \int_{R^+} e^{-ux} p(x,t)dx,$$

then

$$\begin{aligned}\frac{\partial p^L(u,t)}{\partial t} &= -bup^L(u,t) + \int_{R^+} \left[e^{-uy}p^L(u,t) - p^L(u,t)\right.\\ &\quad \left.+ I_{y\in(0,1)} yup^L(u,t)\right]\nu(dy)\\ &= \left\{-b^*u + \int_{R^+} (e^{-uy}-1)\nu(dy)\right\} p^L(u,t).\end{aligned}$$

Solving this Ordinary Differential Equation (ODE), we have

$$\begin{aligned}E[e^{-uX(t)}] &= p^L(u,t)\\ &= p^L(u,0)\exp\left\{t\left[-b^*u + \int_{R^+}(e^{-uy}-1)\nu(dy)\right]\right\}.\end{aligned}$$

Since $p^L(u,0) = 1$, this is consistent with the characteristic function of Lévy subordinators.

Before we use Theorem 1 to derive the Laplace expression for the moments of lifetime T_x as Theorem 2, we introduce an important relation in Lemma 4.

Lemma 4. *Denote*

$$\tilde{Q}(x,t) = -\frac{\partial}{\partial t} R(x,t),$$

where $R(x,t) = \int_0^x p(v,t)dv$, *and*

$$\tilde{Q}_n^{LL}(u,\omega) = (-1)^n \frac{\partial^n \tilde{Q}^{LL}(u,\omega)}{\partial \omega^n},$$

where $\tilde{Q}^{LL}(u,\omega)$ *is the Laplace expression of* $\tilde{Q}(x,t)$. *Let* $M^L(T^n,u)$ *be the Laplace expression of lifetime moments, then*

$$M^L(T^n,u) = \tilde{Q}_n^{LL}(u,0).$$

Proof. Since $\tilde{Q}^{LL}(u,\omega) = \int_{R^+} e^{-\omega t}\tilde{Q}^L(u,t)dt$, we have

$$\begin{aligned}\tilde{Q}_n^{LL}(u,\omega) &= (-1)^n \frac{\partial^n \tilde{Q}^{LL}(u,\omega)}{\partial \omega^n} = (-1)^n \int_{R^+} \frac{\partial^n e^{-\omega t}}{\partial \omega^n} \tilde{Q}^L(u,t)dt\\ &= \int_{R^+} t^n e^{-\omega t}\tilde{Q}^L(u,t)dt.\end{aligned}$$

And as

$$M(T^n,x) = \int_{R^+} t^n \tilde{Q}(x,t)dt,$$

we obtain

$$M^L(T^n,u) = \int_{R^+} t^n \tilde{Q}^L(u,t)dt = \tilde{Q}_n^{LL}(u,0).$$

□

Theorem 2. *For degradation with random jumps described by a Lévy subordinator, the Laplace expression of lifetime moments is*

$$M^L\left(T_X^n,u\right) = n!u^{-1}\left\{b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\right\}^{-n},$$

where $b^* \geq 0$, ν *is the Lévy measure.*

Proof. The Laplace transform of $\tilde{Q}(x,t)$ w.r.t. t is

$$\begin{aligned}\tilde{Q}^L(x,\omega) &= -\int_{R^+} e^{-\omega t}\frac{\partial}{\partial t}R_X(x,t)dt\\ &= h(x) - \omega R_X^L(x,\omega).\end{aligned} \tag{10}$$

For Equation (10), we perform a Laplace transform w.r.t. x on both sides; then

$$\tilde{Q}^{LL}(u,\omega) = -\omega R_X^{LL}(u,\omega) + u^{-1}.$$

From Theorem 1, we have

$$\tilde{Q}^{LL}(u,\omega) = -\omega u^{-1}\left\{\omega + b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\right\}^{-1} + u^{-1}.$$

From Lemma 4:

$$M^L\left(T_X^n,u\right) = \tilde{Q}_n^{LL}(u,0) = (-1)^n\left[\frac{\partial^n \tilde{Q}^{LL}(u,\omega)}{\partial\omega^n}\right]_{\omega=0},$$

where

$$\begin{aligned}&\left[\frac{\partial^n \tilde{Q}^{LL}(u,\omega)}{\partial\omega^n}\right]_{\omega=0}\\ &= -u^{-1}\left[\omega\frac{\partial^n\{\omega + b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\}^{-1}}{\partial\omega^n}\right]_{\omega=0}\\ &\quad -u^{-1}\left[n\frac{\partial^{n-1}\{\omega + b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\}^{-1}}{\partial\omega^{n-1}}\right]_{\omega=0},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^{n-1}\{\omega + b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\}^{-1}}{\partial\omega^{n-1}}\\ &= (-1)^{n-1}(n-1)!\{\omega + b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\}^{-n}.\end{aligned}$$

Therefore, we have

$$M^L\left(T_X^n,u\right) = n!u^{-1}\left\{b^*u - \int_{R^+}(e^{-uy}-1)\nu(dy)\right\}^{-n}.$$

□

3.2. Results based on non-Gaussian OU processes

For degradation with random jumps described by the non-Gaussian OU process $Y(t)$, we derive the explicit expressions of $R_Y(y,t)$ and lifetime moments $M(T_Y^n,y)$ in terms of a Laplace transform, represented by Lévy measures. The results are presented in Theorems 3 and 4.

Theorem 3. *For degradation with random jumps described by a non-Gaussian OU process* $Y(t)$, *the Laplace expression of reliability function is*

$$R_Y^{LL}(u,\omega) = -u^{-1}\int_u^\infty e^{F(v,u,\omega)}g(v)dv,$$

where $F(v,u,\omega) = \int_v^u f(v',\omega)dv'$, $f(v,\omega) = (\omega + b^*v - \int_{R^+}(e^{-vz}-1)\nu(dz))/\alpha v$, *and* $g(v) = -1/\alpha v$. *In addition,* $b^* \geq 0$, ν *is the Lévy measure.*

Proof. Let $p(y,t)$ be the probability density function of $Y(t)$. Based on Sun and Duan (2012), the Fokker–Planck equation for $Y(t)$ is

$$\frac{\partial p(y,t)}{\partial t} = -\alpha\frac{\partial y p(y,t)}{\partial y} - b\frac{\partial p(y,t)}{\partial y} + \int_{R^+}\left[p(y-z,t) - p(y,t) + I_{z\in(0,1)}z\frac{\partial p(y,t)}{\partial y}\right]\nu(dz). \quad (11)$$

For Equation (11), we perform a Laplace transform of $p(y,t)$ w.r.t. t for both sides:

$$\omega p^L(y,\omega) - p(y,0) = -\alpha\frac{\partial y p^L(y,\omega)}{\partial y} - b\frac{\partial p^L(y,\omega)}{\partial y} + \int_{R^+}\left[p^L(y-z,\omega) - p^L(y,\omega) + I_{z\in(0,1)}z\frac{\partial p^L(y,\omega)}{\partial y}\right]\nu(dz). \quad (12)$$

For Equation (12), we perform a Laplace transform of $p^L(y,\omega)$ w.r.t. y for both sides; then

$$\omega p^{LL}(u,\omega) - 1 = \alpha u\frac{\partial p^{LL}(u,\omega)}{\partial u} - bup^{LL}(u,\omega) + \int_{R^+}\left[e^{-uz}p^{LL}(u,\omega) - p^{LL}(u,\omega) + I_{z\in(0,1)}zup^{LL}(u,\omega)\right]\nu(dz).$$

Let $b^* = b - \int_{0<z<1} z\nu(dz)$. We have that

$$\alpha u\frac{\partial p^{LL}(u,\omega)}{\partial u} = \left\{\omega + b^*u - \int_{R^+}\left(e^{-uz}-1\right)\nu(dz)\right\}p^{LL}(u,\omega) - 1.$$

Let $f(u,\omega) = (\omega + b^*u - \int_{R^+}(e^{-uz}-1)\nu(dz))/\alpha u$, and $g(u) = -1/\alpha u$. We have that

$$\frac{\partial p^{LL}(u,\omega)}{\partial u} = f(u,\omega)p^{LL}(u,\omega) + g(u),$$

with $p^{LL}(\infty,\omega) = 0$. By solving this ODE, we have that

$$p^{LL}(u,\omega) = -e^{-\int_u^\infty f(v',\omega)dv'}\int_u^\infty e^{\int_v^\infty f(v',\omega)dv'}g(v)dv = -\int_u^\infty e^{F(v,u,\omega)}g(v)dv,$$

where $F(v,u,\omega) = \int_v^u f(v',\omega)dv'$. Then based on Lemma 3, we have $R_Y^{LL}(u,\omega) = -u^{-1}\int_u^\infty e^{F(v,u,\omega)}g(v)dv$. □

Remark 2. For Equation (11), we perform a Laplace transform of $p(y,t)$ w.r.t. y for both sides:

$$E[e^{-uY(t)}] = p^L(u,t) = \int_{R^+} e^{-uy}p(y,t)dy.$$

Then

$$\frac{\partial p^L(u,t)}{\partial t} = \alpha u\frac{\partial p^L(u,t)}{\partial u} - bup^L(u,t) + \int_{R^+}[e^{-uz}p^L(u,t) - p^L(u,t) + I_{z\in(0,1)}zup^L(u,t)]\nu(dz) = \alpha u\frac{\partial p^L(u,t)}{\partial u} + \left\{-b^*u + \int_{R^+}\left(e^{-uz}-1\right)\nu(dz)\right\}p^L(u,t).$$

By using the method of characteristics to solve this first-order PDE, we have that

$$E[e^{-uY(t)}] = p^L(ue^{\alpha t},0) \times \exp\left\{\int_0^t\left[-b^*ue^{\alpha r} + \int_{R^+}\left(e^{-ue^{\alpha r}y}-1\right)\nu(dy)\right]dr\right\}.$$

Since $p^L(ue^{\alpha t},0) = 1$, we have that

$$E[e^{-uY(t)}] = \exp\left\{\int_0^t\left[-b^*ue^{\alpha r} + \int_{R^+}\left(e^{-ue^{\alpha r}y}-1\right)\nu(dy)\right]dr\right\}.$$

We use Theorem 3 to derive the transform expression for the moments of lifetime T_y in Theorem 4.

Theorem 4. *For degradation with random jumps described by a non-Gaussian OU process $Y(t)$, the Laplace expression of lifetime moments is*

$$M^L\left(T_Y^n,u\right) = (-1)^n u^{-1}n\alpha^{1-n}\int_u^\infty (\ln u - \ln v)^{n-1}e^{F(v,u)}g(v)dv,$$

where $F(v,u) = \int_v^u f(v')dv'$, $f(v) = (b^*v - \int_{R^+}(e^{-vz}-1)\nu(dz))/\alpha v$, and $g(v) = -1/\alpha v$. In addition, $b^* \geq 0$, ν is the Lévy measure.

Proof. The Laplace transform of $\tilde{Q}(y,t)$ w.r.t. t is

$$\tilde{Q}^L(y,\omega) = -\int_{R^+} e^{-\omega t}\frac{\partial}{\partial t}R_Y(y,t)dt = h(y) - \omega R_Y^L(y,\omega). \quad (13)$$

For Equation (13), we perform a Laplace transform w.r.t. y on both sides; then

$$\tilde{Q}^{LL}(u,\omega) = -\omega R_Y^{LL}(u,\omega) + u^{-1}.$$

From Theorem 3, we have that

$$\tilde{Q}^{LL}(u,\omega) = u^{-1}\omega\int_u^\infty e^{F(v,u,\omega)}g(v)dv + u^{-1}.$$

From Lemma 4:

$$M^L\left(T_Y^n,u\right) = \tilde{Q}_n^{LL}(u,0) = (-1)^n\left[\frac{\partial^n\tilde{Q}^{LL}(u,\omega)}{\partial\omega^n}\right]_{\omega=0},$$

where

$$\begin{aligned}\left[\frac{\partial^n \tilde{Q}^{LL}(u,\omega)}{\partial\omega^n}\right]_{\omega=0} &= u^{-1}\left[\frac{\partial^n\left(\omega\int_u^\infty e^{F(\upsilon,u,\omega)}g(\upsilon)d\upsilon\right)}{\partial\omega^n}\right]_{\omega=0}\\ &= u^{-1}\left[\omega\frac{\partial^n\left(\int_u^\infty e^{F(\upsilon,u,\omega)}g(\upsilon)d\upsilon\right)}{\partial\omega^n}\right]_{\omega=0}\\ &\quad + u^{-1}n\left[\frac{\partial^{n-1}\left(\int_u^\infty e^{F(\upsilon,u,\omega)}g(\upsilon)d\upsilon\right)}{\partial\omega^{n-1}}\right]_{\omega=0},\end{aligned}$$

and

$$\begin{aligned}&\left[\frac{\partial^{n-1}\left(\int_u^\infty e^{F(\upsilon,u,\omega)}g(\upsilon)d\upsilon\right)}{\partial\omega^{n-1}}\right]_{\omega=0}\\ &= \left[\int_u^\infty \frac{\partial^{n-1}e^{F(\upsilon,u,\omega)}}{\partial\omega^{n-1}}g(\upsilon)d\upsilon\right]_{\omega=0}\\ &= \left[\int_u^\infty\left(\int_\upsilon^u \frac{1}{\alpha\upsilon'}d\upsilon'\right)^{n-1} e^{F(\upsilon,u,\omega)}g(\upsilon)d\upsilon\right]_{\omega=0}\\ &= \alpha^{1-n}\int_u^\infty (\ln u - \ln\upsilon)^{n-1}\, e^{F(\upsilon,u)}g(\upsilon)d\upsilon.\end{aligned}$$

Therefore, we have that

$$M^L\left(T_Y^n, u\right) = (-1)^n u^{-1} n\alpha^{1-n}\int_u^\infty (\ln u - \ln\upsilon)^{n-1}\, e^{F(\upsilon,u)}g(\upsilon)d\upsilon.$$

□

4. Numerical examples

To illustrate our models, we use an interesting Lévy measure:

$$\nu(dx) = \frac{\kappa}{\Gamma(1-\kappa)}\frac{1}{x^{\kappa+1}}dx,$$

where $x > 0$, $0 < \kappa < 1$, which represents a positive stable process $PS(\kappa)$, a Lévy subordinator, whose distribution is in general unknown in closed form (Barndorff-Nielsen and Shephard, 2012). Notice that if κ is close to zero, the process propagates with big jumps, and if κ is close to one, the process evolves with small jumps. The distribution of this variable is asymmetric and heavy-tailed; i.e., it does not have moments of order κ and above.

When the degradation evolution can be described by this positive stable process, the Laplace expression of reliability function based on Theorem 1 is

$$R_X^{LL}(u,\omega) = u^{-1}\{\omega + u^\kappa\}^{-1}.$$

Based on Theorem 2, the Laplace expression of lifetime moments is

$$M^L\left(T_X^n, u\right) = n!u^{-n\kappa-1}.$$

When the evolution of the degradation can be described by the non-Gaussian OU process driven by $PS(\kappa)$, the Laplace expression of reliability function based on Theorem 3 is

$$R_Y^{LL}(u,\omega) = \alpha^{-1}u^{\alpha^{-1}\omega-1}e^{\alpha^{-1}\frac{1}{\kappa}u^\kappa}\int_u^\infty \upsilon^{-(\alpha^{-1}\omega+1)}e^{-\alpha^{-1}\frac{1}{\kappa}\upsilon^\kappa}d\upsilon.$$

Table 1. Parameter values.

Parameter	*Value*	*Parameter*	*Value*
x; y	[0,30]	κ	0.9
α	0.1		

Based on Theorem 4, the Laplace expression of lifetime moments is

$$\begin{aligned}M^L(T_Y^n,u) &= (-1)^n u^{-1}n\alpha^{1-n}\\ &\quad\times\int_u^\infty (\ln u - \ln\upsilon)^{n-1}\, e^{\alpha^{-1}\frac{1}{\kappa}(u^\kappa-\upsilon^\kappa)}(-\alpha^{-1}\upsilon^{-1})d\upsilon\\ &= u^{-1}n\alpha^{-n}\sum_{i=0}^{n-1}C_{n-1}^i(-1)^i(\ln u)^i e^{\alpha^{-1}\frac{1}{\kappa}u^\kappa}\\ &\quad\times\int_u^\infty (\ln\upsilon)^{n-1-i}\upsilon^{-1}e^{-\alpha^{-1}\frac{1}{\kappa}\upsilon^\kappa}d\upsilon.\end{aligned}$$

The specific values for the parameters are given in Table 1. Sample paths of $X_s(t)$ and $Y(t)$ are shown in Fig. 1 (Lévy subordinator b) and Fig. 2 (non-Gaussian OU b), respectively. The system fails when the degradation exceeds the respective failure threshold. The inversion algorithms for Laplace transform (Abate and Whitt, 1995; Brančík, 2007) were implemented to invert Laplace expressions in order to compute the values of reliability and lifetime moments.

Figure 3 and Figure 4 show the reliability with respect to time t and the failure threshold for $X_s(t)$ and $Y(t)$, respectively. The reliability decreases as the time increases, and it increases as the threshold increases. Figure 5 shows the reliability with respect to time t when the failure thresholds are 15 and 20, respectively. The reliability based on $Y(t)$ decreases faster than that based on $X_s(t)$. Figure 6 illustrates the first moment of the lifetime with respect to the failure threshold. The mean failure time based on $Y(t)$ is less than that based on $X_s(t)$ for the same threshold. These observations correspond to the evolution of $Y(t)$ and $X_s(t)$; the mean path of $Y(t)$ is exponential with respect to time t, whereas the mean path of $X_s(t)$ is linear with respect to time t. In addition to the Lévy measure used in this example, we can

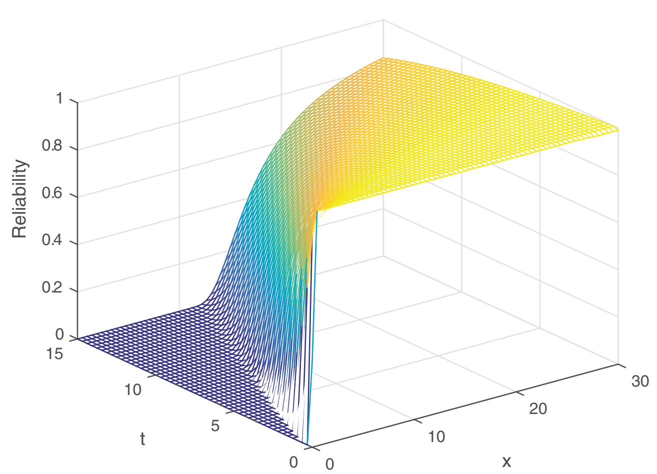

Figure 3. Reliability function with respect to time t and failure threshold x based on $X_s(t)$.

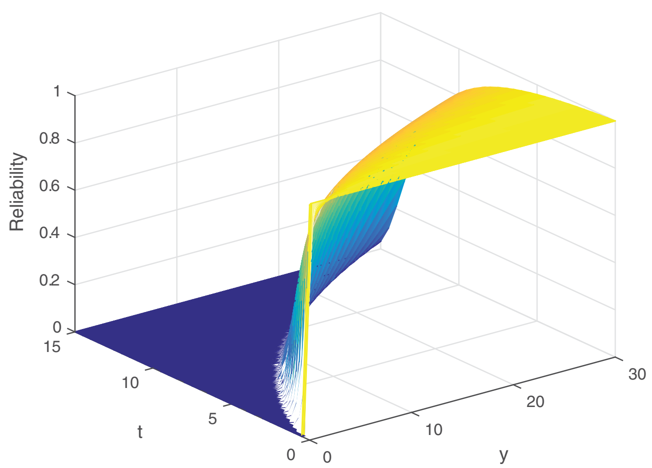

Figure 4. Reliability function with respect to time t and failure threshold y based on $Y(t)$.

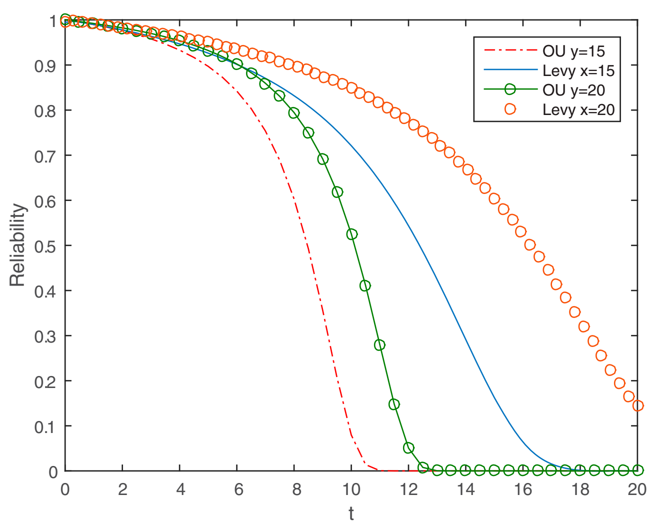

Figure 5. Reliability function with respect to time t based on $X_s(t)$ and $Y(t)$.

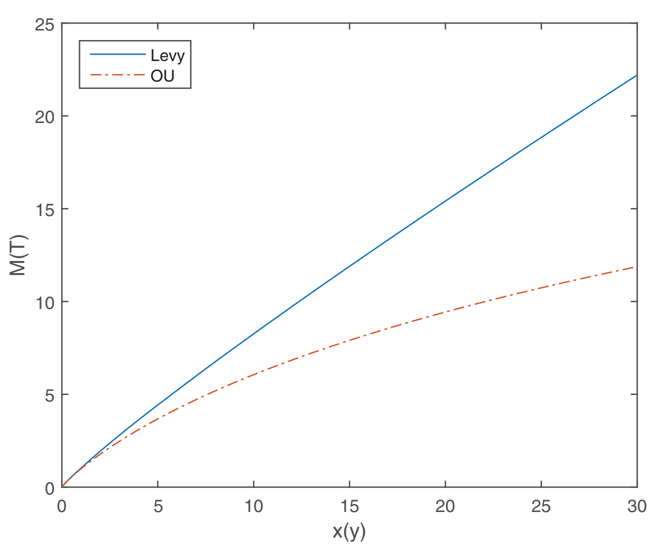

Figure 6. First moment of lifetime with respect to failure threshold based on $X_s(t)$ and $Y(t)$.

specify different Lévy measures to fit the corresponding degradation data, in order to construct models and analyze reliability and lifetime.

5. Conclusions

In this article, we presented novel models concerning the stochastic mechanism of a complex degradation process that is also subjected to random jumps. Based on the Fokker–Planck equation, we derived explicit results for reliability function and lifetime moments in terms of a Laplace transform. The Laplace expressions of the reliability function and lifetime moments are represented by Lévy measures. Our model is general, as we can specify many different Lévy measures to handle many different kinds of degradation data sets. The models in the literature become special cases of our models.

The new method provides a convenient and general way to evaluate system reliability. When the degradation data are available, our results are explicit and compact enough for effective and efficient statistical inference on lifetime characteristics. The reliability estimation is expected to be more accurate, as our models integrally consider all the stylized features of degradation data series including temporal uncertainty, jumps, independence/dependence, linearity/nonlinearity, symmetry/asymmetry, and light/heavy tails. Based on the precise reliability estimation and prediction, an appropriate and valuable maintenance policy can be proposed and implemented.

One of the challenging aspects in reliability analysis is how to formulate reliability functions for degradation processes under dynamic environments. Our model based on Fokker–Planck equations provides a new methodology to overcome this challenge. We will focus on deriving Fokker–Planck equations for degradation processes under dynamic environments in our future work. In order to apply the model to degradation data analysis, statistical inference on Lévy measures is another potential research topic. Traditional maximum likelihood estimation and Bayesian estimation are not convenient for such general jump processes without closed-form distributions. To apply our models to real degradation dataset, the parametric estimation for subordinators and OU processes in Jongbloed and van der Meulen (2006) can be explored.

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Notes on contributors

Yin Shu is currently a Ph.D. candidate in Industrial Engineering at the University of Houston, Houston, Texas. His research interests are mainly focused on applied probability theory and statistics in Lévy-based stochastic processes, related mathematical methodology, and data analytics, with applications in reliability/safety engineering. He is a member of INFORMS and IIE.

Qianmei Feng is an Associate Professor and the Brij and Sunita Agrawal Faculty Fellow in the Department of Industrial Engineering at the University of Houston. She received a Ph.D. degree in Industrial Engineering from the University of Washington, Seattle, Washington, in 2005. She has focused her research on the areas of system modeling, analysis, and optimization in quality and reliability engineering, with applications in evolving technologies (e.g., MEMS, biomedical implant devices), homeland security, and healthcare. Her research has been supported by the National Science Foundation, Department of Homeland Security, Texas Department of Transportation, and Texas Higher Education Coordinating Board. She served as the president for the Division of Quality Control & Reliability Engineering in the Institute of Industrial Engineers between 2009 and 2010. She is also a member of INFORMS, ASQ, and Alpha Pi Mu.

Edward P.C. Kao is a Professor of Mathematics and the Graduate Program Director of Financial Mathematics at the University of Houston, Houston, Texas, since 2002. He received his Ph.D. degree in Industrial Engineering from Stanford University and did his post-doctoral work at Yale University. Between September 1997 and May 1998, he was a visiting scientist at Massachusetts Institute of Technology. He taught at Yale University and the University of Wisconsin at Milwaukee and joined the University of Houston in 1976 as an Associate Professor in Operations Research. He was the Area Editor in Computational Probability and Analysis of *INFORMS Journal on Computing*, 1996–2007, and served as Associate Editor of *Operations Research*, 1983–1987. He is the author of a book entitled *An Introduction to Stochastic Processes*, published by Duxbury Press.

Hao Liu is a tenured Associate Professor in Biostatistics at the Dan L. Duncan NCI-designated Comprehensive Cancer Center of Baylor College of Medicine in Houston Texas. He earned his Ph.D. in Biostatistics from the University of Washington at Seattle. He is a National Institute of Health–funded investigator with over 35 active and completed research grants. His current research areas include survival analysis and clinical trial designs.

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