Math 4370, Fall 2003, Solutions to HW#4, E. Kao

12.24 \( S_0 = 50, \mu = 0.18, \sigma = 0.30, T = 2 \). Recall

\[
\ln S_T \sim \phi(\ln S_0 + \alpha T, \sigma \sqrt{T})
\]

In this case,

\[
\begin{align*}
\alpha &= \mu - \frac{\sigma^2}{2} = 0.18 - \frac{0.3^2}{2} = 0.135 \\
\ln S_0 + \sigma T &= \ln(50) + 0.135(2) = 4.182 \\
\sigma \sqrt{T} &= 0.3 \sqrt{2} = 0.42426
\end{align*}
\]

Thus

\[
\ln S_T \sim \phi(4.182, 0.42426)
\]

The probability distribution for the stock price follows the lognormal distribution. The mean and standard deviation of the stock price are:

\[
\begin{align*}
E(S_T) &= S_0 e^{\mu T} = 50 e^{0.18(2)} = 71.67 \\
Var(S_T) &= S_0^2 e^{2 \mu T} e^{\sigma^2 T} - 1 = (50)^2 e^{2(0.18)(2)} e^{0.09(2)} - 1 = 1012.9 \\
SDV(S_T) &= \sqrt{1012.9} = 31.83
\end{align*}
\]

The 95% "confidence interval" is

\[
4.182 + 1.96(0.42426) = 5.0135 \\
4.182 - 1.96(0.42426) = 3.3505
\]

or

\[
e^{3.3505} < S_T < e^{5.0135} \quad \text{or} \quad 28.52 < S_T < 150.43 \quad \square
\]

12.26 We need to find the expected value of the contingent claim \( f \) at time \( T \), namely, \( E[S_T^2] \). But we do not have to find it directly (cf. the hint given in the text). From (12.4) and (12.5), we note

\[
E[S_T^2] = Var[S_T] + E^2[S_T] = S_t^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1] - \left(S_t e^{-\mu(T-t)}\right)^2
\]

In a risk-neutral world, we set \( \mu \) to \( r \) and write

\[
\hat{E}[S_T^2] = S_t^2 e^{(2r+\sigma^2)(T-t)}.
\]

Now we find the price of the contingent claim at time \( t \):

\[
\begin{align*}
\hat{E}[S_T^2] &= S_t^2 e^{(2r+\sigma^2)(T-t)} \\
\hat{E}[S_T^2] &= S_t^2 e^{(2r+\sigma^2)(T-t)}
\end{align*}
\]

\[
\hat{E}[S_T^2] = S_t^2 e^{(2r+\sigma^2)(T-t)} = S_A^2
\]

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where we let $A = e^{(r + \sigma^2)(T - t)}$ and use $S$ to denote $S_t$ for simplicity.

(b) To verify the above $f$ satisfies (12.15), we find

$$
\frac{\partial f}{\partial t} = -S^2 e^{(r + \sigma^2)(T - t)} (r + \sigma^2) = -S^2 (r + \sigma^2) A
$$

$$
\frac{\partial f}{\partial S} = 2S e^{(r + \sigma^2)(T - t)} = 2SA \quad \frac{\partial^2 f}{\partial S^2} = 2e^{(r + \sigma^2)(T - t)} = 2A
$$

Now we write the left side of the BLS-PDE (12.15)

$$
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} = -S^2 (r + \sigma^2) A + rS (2SA) + \frac{1}{2} \sigma^2 S^2 (2A)
$$

$$
= -rS^2 A - \sigma^2 S^2 A + 2rS^2 A + \sigma^2 S^2 A = rS^2 A
$$

The last term is just $rf = rS^2 A$. So we conclude that the $f$ so derived satisfies (12.15). 

12.27 The stock is non-dividend paying. We are given with $S_0 = 30$, $X = 29$, $r = 0.05$, $\sigma = 0.25$, $T = 4/12$. We see

$$
d_1 = \frac{\ln \left( \frac{S_0}{X} \right) + (r + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}}
$$

$$
= \frac{\ln \left( \frac{30}{29} \right) + (0.05 + \frac{0.25^2}{2}) (1/3)}{0.25 \sqrt{1/3}} = 0.4225
$$

and

$$
d_2 = d_1 - \sigma \sqrt{T} = 0.4225 - 0.25 \sqrt{1/3} = 0.2782
$$

We find from the table of normal distribution

$$
\Phi(d_1) = \Phi(0.4225) = 0.6637 \quad \Phi(-d_1) = \Phi(-0.4225) = 0.3363
$$

$$
\Phi(d_2) = \Phi(0.2782) = 0.6096 \quad \Phi(-d_2) = \Phi(-0.2782) = 0.3904
$$

(a) The price of a European call is

$$
c = S_0 \Phi(d_1) - X e^{-rT} \Phi(-d_2) = 30(0.6637) - (29)e^{-0.05(1/3)}(0.6096) = \$$

$$
2.52
$$

(b) The American call price is the same as the European call. It is $2.52$.

(c) The price of a European put is

$$
p = X e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1) = 29e^{-0.05(1/3)}(0.3904) - 30(0.3363) = 1.05.
$$

(d) The put-call parity says:

$$
p + S_0 = c + X e^{-rT}
$$

and so we check

$$
LHS = p + S_0 = 1.05 + 30 = 31.05
$$
and
\[ \text{RHS} = c + X e^{-rT} = 2.52 + 29e^{-0.05(1/3)} = 31.04 \]
The minor discrepancy is due to round-off error. □

12.28 The dividend payment induces a reduction of \( S_0 \). We now reset the value of \( S_0 \) as follows:
\[ S_0 = 30 - e^{-rT}(0.5) = 30 - (0.5)e^{-0.05(1.5/12)} = 29.5031 \]
The rest is standard.

\[ d_1 = \frac{\ln \left( \frac{S_0}{X} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = \frac{\ln \left( \frac{29.5031}{29} \right) + \left( 0.05 + \frac{(0.25)^2}{2} \right)(1/3)}{0.25 \sqrt{1/3}} = 0.3068 \]

and
\[ d_2 = d_1 - \sigma \sqrt{T} = 0.3068 - 0.25 \sqrt{1/3} = 0.1625 \]
We find from the table of normal distribution
\[ \Phi(d_1) = \Phi(0.3068) = 0.6205 \quad \Phi(-d_1) = \Phi(-0.3068) = 0.3795 \]
\[ \Phi(d_2) = \Phi(0.1625) = 0.5645 \quad \Phi(-d_2) = \Phi(-0.1625) = 0.4355 \]

(a) The price of a European call is
\[ c = S_0 N(d_1) - X e^{-rT} N(d_2) = 29.5031(0.6205) - (29)e^{-0.05(1/3)}(0.5645) = 2.21 \]

(b) The price of a European put is
\[ p = X e^{-rT} N(-d_2) - S_0 N(-d_1) = 29e^{-0.05(1/3)}(0.4355) - 29.5031(0.3795) = 1.22 \]

(c) Assume that the stock price is sufficiently high (i.e., it is deep in-the-money) so that you will either exercise the call before the ex-dividend date or you will wait until maturity and then exercise. If you exercise immedicately before the ex-dividend date, then you collect 50\( \$ \) at \( t = 1.5 \). If you wait until \( t = 4 \) and then exercise, you save yourself some time-value of the money. The PV at time \( t = 1.5 \) of this saving is
\[ 29(1 - e^{-0.05(2.5/12)}) = 0.3005 \]
or 30\( \$ \). Thus if the stock price is sufficiently high, you should exercise early (i.e., immideately before the ex-dividend date) to get the 20\( \$ \) saving. □