6.13 Let $Y = \log(X)$. Thus $X = e^Y$ and $dy = (1/x)dx$, $dx = e^y dy$. A change of variable gives

$$f_Y(y) = \alpha((e^y)^{\alpha-1}e^{-\alpha e^y})(e^y) = \alpha(e^y e^{-\alpha y}) = \alpha(\exp(\alpha y - e^{\alpha y})$$

$$= \frac{1}{1/\alpha} \exp\left(y/(1/\alpha) - e^{y/(1/\alpha)}\right) \quad -\infty < y < \infty$$

Let $Y = (1/\alpha)Z$, then $dy = (1/\alpha)dz$ and

$$f_Z(z) = \exp(z - e^z) \quad -\infty < z < \infty \quad (1)$$

We see that the family of distributions of $Y_i$ is a scale family with scale parameter $1/\alpha$. By Theorem 3.5.6, we can write $Y_i = (1/\alpha)Z_i$, where $Z_i$ follows the density (1). Now

$$S(X) = \frac{\log X_1}{\log X_2} = \frac{Y_1}{Y_2} = \frac{(1/\alpha)Z_1}{(1/\alpha)Z_2} = \frac{Z_1}{Z_2}$$

Clearly, $S(X)$ is independent of $\alpha$. So $\log X_1/\log X_2$ is an ancillary statistic. \[\square\]

6.19 We use Definition 6.2.21 on p. 285 to check the completeness of $X$. We check if $E_p(g(X)) = 0$ for all $p$ implies $P(g(X) = 0) = 1$ for all $g$.

**Distribution 1:** Here

$$E_p(g(X)) = \sum_{i=0}^{2} g(x)P(X = x) = g(0)p + g(1)3p + g(2)(1 - 4p)$$

The above is a polynomial of degree 1 in $p$. So we require $g(2) = 0$. This implies $g(0) + 3g(1) = 0$ and $g(0) = -3g(1)$. Thus the expectation is zero for all $p$ but $g(x)$ need not be identically zero. Hence the family of distribution for $X$ is not complete.

**Distribution 2:** Here

$$E_p(g(X)) = \sum_{i=0}^{2} g(x)P(X = x) = g(0)p + g(1)p^2 + g(2)(1 - p - p^2)$$

We see that the above is a polynomial of degree 2 in $p$. To make sure the right side is zero, we require that each coefficient must be zero. This implies that $g(2) = g(0) = g(1) = 0$. This means that the family of distribution for $X$ is complete. \[\square\]

6.22 (a) The joint density of the sample is

$$\prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \quad (2)$$
Hence $\prod_{i=1}^{n} X_i$ is a sufficient statistic but not $\sum_{i=1}^{n} X_i$.

(b) We write (2) as

$$\prod_{i=1}^{n} f(x_i|\theta) = \theta^n \exp \left( (\theta - 1) \log \left( \prod_{i=1}^{n} x_i \right) \right)$$

A comparison of the above with (6.2.7) clearly shows that the joint density is in the exponential family. By Theorem 6.2.25, we conclude

$$\log \left( \prod_{i=1}^{n} x_i \right)$$

is a complete and sufficient statistic. Since the above is a one-to-one function of $\prod_{i=1}^{n} x_i$, we conclude that the latter is a complete sufficient statistic for $\theta$. □

6.30 (a) From our HW3, Problem 6.9 (b), we know that $X_{(1)}$ is a minimal sufficient statistic for $\mu$. First we need to find the density for $X_{(1)}$ and then use Definition 6.2.21 to check for completeness. Recall

$$f_{X_{(1)}}(x) = \left( \begin{array}{c} n \\ n - 1 \end{array} \right) (1 - F_X(x))^{n-1} f_X(x) = n \left( e^{-x-\mu} \right)^{n-1} e^{-(x-\mu)}$$

For notational convenience, we define $Y \equiv X_{(1)}$ and hence

$$f_Y(y) = ne^{-n(y-\mu)} \quad y > \mu$$

Now we are ready for applying Definition 6.2.21. Note

$$E_\mu(g(Y)) = \int_{\mu}^{\infty} g(y)ne^{-n(y-\mu)} dy = ne^{n\mu} \int_{\mu}^{\infty} g(y)e^{-ny} dy.$$ 

Clearly, $ne^{n\mu} > 0$ for all $\mu$ and does not depend on $y$. If we require $E_\mu(g(Y)) = 0$, this means we require

$$\int_{\mu}^{\infty} g(y)e^{-ny} dy = 0$$

for all $\mu$. This further implies

$$\frac{d}{d\mu} \int_{\mu}^{\infty} g(y)e^{-ny} dy = -g(\mu)e^{-n\mu} = 0$$

where the first equality holds by an application of the fundamental theorem of calculus. The last equality implies that we must have $g(\mu) = 0$ for all $\mu$. This means $X_{(1)}$ is complete.
(b) Since we have shown that \( X_{(1)} \) is a complete sufficient statistic for \( \mu \), by Basu’s Theorem, \( X_{(1)} \) is independent of any ancillary statistic. If we can show that \( S^2 \) is ancillary, i.e., \( S^2 \) follows a distribution that is independent of \( \mu \), then we are done.

We observe that \( f(x|\mu) \) is a location family. So we can write \( X_i = Z_i + \mu \), where \( Z_i \sim f(x|0) \). Thence

\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{1}{n-1} \sum_{i=1}^{n} ((Z_i + \mu) - (\overline{Z} + \mu))^2 \\
= \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \overline{Z})^2.
\]

Thus \( S^2 \) is function of \( Z_1, \ldots, Z_n \) and independent of \( \mu \). This means \( S^2 \) is ancillary. By Basu’s Theorem, \( S^2 \) is independent of \( X_{(1)} \). \( \square \)