9.3 (a) In Exercise 7.10, we found that for any fixed $\alpha$, the MLE for $\beta$ is $X_{(n)}$. Since $\beta$ is a scale parameter, $X_{(n)}/\beta$ is a pivot. So

$$0.05 = P_\beta \left( \frac{X_{(n)}}{\beta} \leq c \right) = P_\beta \left( \text{all } X_i \leq c\beta \right) = \left( \frac{c\beta}{\beta} \right)^{\alpha n} = c^{\alpha n}.$$ 

This implies

$$c = (0.05)^{\frac{1}{\alpha n}}.$$ 

Therefore, we conclude

$$0.95 = P_\beta \left( \frac{X_{(n)}}{\beta} > c \right) = P_\beta \left( \frac{\beta}{X_{(n)}} < \frac{1}{c} \right) = P_\beta \left( \beta < \frac{X_{(n)}}{c} \right)$$

and the 95% upper confidence limit for $\beta$ is given by

$$\left\{ \beta : \beta < \frac{X_{(n)}}{(0.05)^{\frac{1}{\alpha n}}} \right\}.$$ 

(b) In Exercise 7.10, we found $\hat{\alpha} = 12.59$ and $X_{(14)} = 25$. Thus

$$c = (0.05)^{\frac{1}{\alpha n}} = (0.05)^{\frac{1}{12.59 \cdot 14}} = 0.98315$$

and so

$$\left\{ \beta : \beta < \frac{25}{0.98315} \right\} = \{ \beta : \beta < 25.43 \}$$

i.e, the 95% upper confidence interval for $\beta$ is given by $(0, 25.43)$. \(\square\)

9.13 (a) $X \sim \text{beta}(\theta, 1)$ and hence

$$f(x|\theta, 1) = \theta x^{\theta-1}, \quad 0 < x < 1$$

Now $Y = -(\log X)^{-1}$ implies $X = \exp(-1/Y)$ and $dx = (1/y^2) \exp(-1/y) dy$. Thus the density for $Y$ is given by

$$f_Y(y) = \frac{\theta}{y^2} e^{-\frac{\theta}{y}}, \quad y > \infty$$

Hence

$$P \left( \frac{Y}{2} < \theta < Y \right) = P(\theta < Y < 2\theta)$$

$$= \int_{\theta}^{2\theta} \frac{\theta}{y^2} e^{-\frac{\theta}{y}} dy = \left[ e^{-\frac{\theta}{y}} \right]_{\theta}^{2\theta} = e^{-0.5} - e^{-1} = 0.23865.$$ 

(b) Let $T = X^\theta$. A change of variable from $X$ to $T$ will show that the density of $T$ is uniform over $(0, 1)$. We form a pivotal interval with

$$P(a < X^\theta < b) = b - a$$
or equivalently
\[
P \left( \frac{\log a}{\log x} > \theta > \frac{\log b}{\log x} \right) = P \left( \frac{\log b}{\log x} < \theta < \frac{\log a}{\log x} \right) = b - a
\]

Since \( X^\theta \sim U(0, 1) \). The interval
\[
\left\{ \theta : \frac{\log b}{\log x} < \theta < \frac{\log a}{\log x} \right\}
\]
will have confidence 0.23865 so long \( b - a = 0.23865 \).

(c) The interval given in Part (a) is a special case of the one given in Part (b). For the one produced in the latter, there are rooms for optimization. We can minimize
\[
\log b - \log a
\]
subject to
\[
b - a = 1 - \alpha,
\]
or \( b = 1 - \alpha + a \). So we want to choose \( a \) so as to minimize the following:
\[
\log(1 - \alpha + a) - \log a = \log \frac{1 - \alpha + a}{a} = \log \left( 1 + \frac{1 - \alpha}{a} \right)
\]
The above is minimized by taking \( a \) as large as possible. But we take \( b = 1 \), then the largest possible value for \( a \) to assume is \( \alpha \). Hence the best \( 1 - \alpha \) pivotal interval is
\[
\left\{ \theta : 0 < \theta < \frac{\log \alpha}{\log x} \right\}.
\]
In conclusion, the interval given in Part (a) is not optimal. A shorter interval with confidence 0.239 is
\[
\left\{ \theta : 0 < \theta < \frac{\log(1 - 0.239)}{\log(x)} \right\} \quad \Box
\]

9.17 (a) \( X_i \sim U \left( \theta - \frac{1}{2}, \theta + \frac{1}{2} \right) \). This is a location family. So we define a pivot quantity
\[
Y_i = X_i - \left( \theta - \frac{1}{2} \right) = X_i - \theta + \frac{1}{2}.
\]
Now \( Y_i \sim (0, 1) \) and thus \( Y_i \) is a pivot (i.e., its density is independent of \( \theta \)). Let \( T = Y_{(n)} \). Then
\[
T = \max_i \left\{ X_i - \theta + \frac{1}{2} \right\} = X_{(n)} - \theta + \frac{1}{2}.
\]
We know that
\[
f_T(t) = nt^{n-1}, \quad 0 < t < 1.
\]
We define $c_1$ and $c_2$ so that
\[ \int_0^{c_1} nt^{n-1}dt = \frac{\alpha}{2} \quad \int_{a_2}^1 nt^{n-1}dt = \frac{\alpha}{2} \]

We solve the above for $c_1$ and $c_2$. This gives
\[ c_1 = \left(\frac{\alpha}{2}\right)^{\frac{1}{n}} \quad c_1 = \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}} \]

This implies that
\[ P(c_1 < T < c_2) = 1 - \alpha \]

or equivalently,
\[ P\left(c_1 < X_{(n)} - \theta + \frac{1}{2} < c_2\right) = 1 - \alpha \]

Thus a $1 - \alpha$ confidence interval is given by
\[ \left\{ \theta : X_{(n)} + \frac{1}{2} - c_2 < \theta < X_{(n)} + \frac{1}{2} - c_1 \right\} \]

(b) $X_i$ follows the density
\[ f(x|\theta) = \frac{2x}{\theta^2} \quad 0 < x < \theta \]

The above is a scale family. We let $Y = X/\theta$ and do a transformation of variable. This gives
\[ f_Y(y) = 2y \quad 0 < y < 1 \]

We see that $Y$ is a pivot (i.e., its density is independent of $\theta$). Let $T = Y_{(n)}$. What is the density of $T$? We see that
\[ P(Y_{(n)} < t) = P(\text{all } Y_i < t) = (t^2)^n = t^{2n} \]

Hence
\[ f_T(t) = 2n(t^{2n-1}) \quad 0 < t < 1 \]

We define $c_1$ and $c_2$ so that
\[ \int_0^{c_1} 2nt^{2n-1}dt = \frac{\alpha}{2} \quad \int_{a_2}^1 2nt^{2n-1}dt = \frac{\alpha}{2} \]

Since
\[ \int 2nt^{2n-1}dt = t^{2n} + C \]

We have
\[ c_1^{2n} = \frac{\alpha}{2} \quad \Rightarrow \quad c_1 = \left(\frac{\alpha}{2}\right)^{\frac{1}{2n}} \]
and similarly
\[ c_2 = \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}}. \]
The above implies
\[ P(c_1 < T < c_2) = 1 - \alpha \]
or
\[ P\left(c_1 < \frac{X(n)}{\theta} < c_2\right) = 1 - \alpha \]
or
\[ P\left(\frac{X(n)}{c_2} < \theta < \frac{X(n)}{c_1}\right) = 1 - \alpha. \]
Therefore a \(1 - \alpha\) confidence interval for \(\theta\) is
\[ \left\{ \theta : \frac{X(n)}{c_2} < \theta < \frac{X(n)}{c_1}\right\} \]

9.23 (a) Let \(Y = \sum_{i=1}^{n} X_i\) as suggested. Then \(Y \sim Poisson(n\lambda)\). Let \(\hat{\lambda}\) be the MLE of \(\lambda\). We know that \(\hat{\lambda} = y/n\). The LRT statistic for \(H_0 : \lambda = \lambda_0\) and \(H_1 : \lambda \neq \lambda_0\) is given by
\[ g(y) = \frac{e^{-n\lambda_0} \left(\frac{n\lambda_0}{y}\right)^y}{e^{-n\hat{\lambda}} \left(\frac{n\hat{\lambda}}{y}\right)^y} = e^{y-n\lambda_0} \left(\frac{n\lambda_0}{y}\right)^y. \]
The acceptance region is given by
\[ A(\lambda_0) = \{y : g(y) > c(\lambda_0)\} \]
where \(c(\lambda_0)\) is chosen such that
\[ P(Y \in A(\lambda_0)) \geq 1 - \alpha. \] (1)

For \(n\lambda = (14)(50) = 700\), we plot \(g(y)\) for \(0 < y < 400\). We see that \(g(y)\)
is a unimodal function of $y$. This can also formally be established. We thus conclude that the acceptance region is an interval. For a fixed $y$, there will be a smallest $\lambda_0$, call it $a(y)$, and a largest $\lambda_0$, call it $b(y)$, such that (1) holds. The confidence interval is then $C(y) = (a(y), b(y))$. This is to be done numerically.

(b) If you use the approach given in Example 9.2.15, the confidence interval is given by $(57.75, 66.49)$. The confidence interval constructed using Part (b) is expected to be similar. 

9.34 (a) A $1 - \alpha$ confidence interval for $\mu$ is given by

$$\left\{ \mu : \bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}} \right\}$$

We require

$$2(1.96)\frac{\sigma}{\sqrt{n}} \leq \frac{\sigma}{4}$$

or

$$\sqrt{n} \geq 4(2)(1.96) \quad \Rightarrow \quad n = 245.9$$

So $n = 246$ is sufficient.

(b) The length of a 95% confidence interval is

$$2t_{n-1, 0.025} \frac{S}{\sqrt{n}}$$

This means we require

$$P \left( 2t_{n-1, 0.025} \frac{S}{\sqrt{n}} \leq \frac{\sigma}{4} \right) \geq 0.9$$

or

$$P \left( 4t_{n-1, 0.025}^2 \frac{S^2}{n} \leq \frac{\sigma^2}{16} \right) \geq 0.9$$

or

$$P \left( \frac{(n-1)S^2}{\sigma^2} \leq \frac{(n-1)n}{64 \times t_{n-1, 0.025}^2} \right) \geq 0.9$$

or

$$P \left( \chi^2_{n-1} \leq \frac{(n-1)n}{64 \times t_{n-1, 0.025}^2} \right) \geq 0.9$$

So we set

$$\chi^2_{n-1, 0.90} \approx \frac{(n-1)n}{64 \times t_{n-1, 0.025}^2}$$

We find that $n = 276$. 