Chapter 3

Second Order Linear Differential Equations

3.1 Introduction; Basic Terminology

Recall that a first order linear differential equation is an equation which can be written in the form

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on some interval \( I \). A second order linear differential equation has an analogous form.

A second order, linear differential equation is an equation which can be written in the form

\[ y'' + p(x)y' + q(x)y = f(x) \] (1)

where \( p, q, \) and \( f \) are continuous functions on some interval \( I \).

The functions \( p \) and \( q \) are called the coefficients of the equation; the function \( f \) on the right-hand side is called the forcing function or the nonhomogeneous term. The term “forcing function” comes from the applications of second-order equations; an explanation of the alternative term “nonhomogeneous” is given below.

A second order equation which is not linear is said to be nonlinear.

**Remarks on “Linear.”** Set \( L[y] = y'' + p(x)y' + q(x)y \). If we view \( L \) as an “operator” that transforms a twice differentiable function \( y = y(x) \) into the continuous function

\[ L[y(x)] = y''(x) + p(x)y'(x) + q(x)y(x), \]

then, for any two twice differentiable functions \( y_1(x) \) and \( y_2(x) \),

\[ L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)] \]
and, for any constant $c$,

$$L[cy(x)] = cL[y(x)].$$

As introduced in Section 2.1, $L$ is a linear operation, specifically, a linear differential operator:

$$L : C^2(I) \to C(I)$$

where $C^2(I)$ is the vector space of twice continuously differentiable functions on $I$ and $C(I)$ is the vector space of continuous functions on $I$.

The first thing we need to know is that an initial-value problem has a solution, and that it is unique.

**THEOREM 1. (Existence and Uniqueness Theorem)** Given the second order linear equation (1). Let $a$ be any point on the interval $I$, and let $\alpha$ and $\beta$ be any two real numbers. Then the initial-value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(a) = \alpha, \quad y'(a) = \beta$$

has a unique solution.

A proof of this theorem is beyond the scope of this course.

**Remark:** Chapter 2 gives a method for finding the general solution of any first order linear equation. In contrast, there is no general method for solving second (or higher) order linear differential equations. There are, however, methods for solving certain special types of second order linear equations and we will consider these in this chapter.

**DEFINITION 1. (Homogeneous/Nonhomogeneous Equations)** The linear differential equation (1) is homogeneous\(^1\) if the function $f$ on the right side is 0 for all $x \in I$. In this case, equation (1) becomes

$$y'' + p(x)y' + q(x)y = 0.$$  \hspace{1cm} (2)

Equation (1) is nonhomogeneous if $f$ is not the zero function on $I$, i.e., (1) is nonhomogeneous if $f(x) \neq 0$ for some $x \in I$.

For reasons which will become clear, almost all of our attention is focused on homogeneous equations.

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\(^1\)This use of the term “homogeneous” is completely different from its use to categorize the first order equation $y' = f(x, y)$ in Section 2.3.
3.2 Homogeneous Equations

As defined above, a second order, linear, homogeneous differential equation is an equation that can be written in the form

\[ y'' + p(x) y' + q(x) y = 0 \]  \hspace{1cm} (H)

where \( p \) and \( q \) are continuous functions on some interval \( I \).

The trivial solution. The first thing to note is that the zero function, \( y(x) = 0 \) for all \( x \in I \), (also denoted by \( y \equiv 0 \)) is a solution of \( (H) \). The zero solution is called the trivial solution. Obviously our main interest is in finding nontrivial solutions. ■

Let \( S = \{ y = y(x) : y \text{ is a solution of } (H) \} \). \( S \) is a subset of \( C^2(I) \), the vector space of twice-continuously differentiable functions.

**THEOREM 1.** Let \( y = u(x), y = v(x) \in S \), and let \( C \) be any real number. Then

\[
\begin{align*}
y(x) &= u(x) + v(x) \in S \text{ and } \vspace{0.2cm} \\
y(x) &= Cu(x) \in S.
\end{align*}
\]

That is, \( S \) is a subspace of \( C^2(I) \).

**PROOF:** Let \( L \) be the linear differential operator: \( L[y] = y'' + p(x)y' + q(x)y \). Since \( u \) and \( v \) are solutions and \( L \) is linear,

\[
L[u(x)] = 0 \quad \text{and} \quad L[v(x)] = 0,
\]

and since \( L \) is linear,

\[
l[u + v] = L[u] + L[v] = 0 + 0 = 0 \quad \text{and} \quad L[Cu] = CL[u] = C \cdot 0 = 0.
\]

Therefore, \( y = u + v \in S \) and \( y = Cu \in S \). Thus, \( S \) is closed with respect to addition and multiplication by a scalar which implies that \( S \) is a subspace of \( C^2(I) \). ■

Theorem 1 can be restated equivalently as:

**THEOREM** If \( y = y_1(x), y = y_2(x) \in S \) and \( C_1, C_2 \) are real numbers, then

\[ y(x) = C_1 y_1(x) + C_2 y_2(x) \in S. \]

**PROOF:** Let \( L[y] = y'' + p(x)y' + q(x)y \). Since \( y_1 \) and \( y_2 \) are solutions and \( L \) is linear,

\[
L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2] = C_1 \cdot 0 + C_2 \cdot 0 = 0 + 0 = 0.
\]

Thus, \( y = C_1 y_1 + C_2 y_2 \in S \). ■
The expression
\[ C_1 y_1 + C_2 y_2 \]
is called a linear combination of \( y_1 \) and \( y_2 \). Thus, Theorem 1 says that any linear combination of solutions of (H) is a solution of (H).

Note that the equation
\[ y(x) = C_1 y_1(x) + C_2 y_2(x), \tag{1} \]
where \( C_1 \) and \( C_2 \) are arbitrary constants, has the form of the general solution of equation (H). So the question is: If \( y_1 \) and \( y_2 \) are solutions of (H), is the expression (1) the general solution of (H)? That is, can every solution of (H) be written as a linear combination of \( y_1 \) and \( y_2 \)? It turns out that (1) may or not be the general solution; it depends on the relationship between the solutions \( y_1 \) and \( y_2 \).

Suppose that \( y = y_1(x) \) and \( y = y_2(x) \) are solutions of equation (H). Under what conditions is (1) the general solution of (H)?

Let \( u = u(x) \) be any solution of (H) and choose any point \( a \in I \). Suppose that \( \alpha = u(a) \), \( \beta = u'(a) \). Then \( u \) is a member of the two-parameter family (1) if and only if there are values for \( C_1 \) and \( C_2 \) such that
\[
C_1 y_1(a) + C_2 y_2(a) = \alpha \\
C_1 y'_1(a) + C_2 y'_2(a) = \beta
\]
If we multiply the first equation by \( y'_2(a) \), the second equation by \( -y_2(a) \), and add, we get
\[ [y_1(a)y'_2(a) - y_2(a)y'_1(a)]C_1 = \alpha y'_2(a) - \beta y_2(a). \]
Similarly, if we multiply the first equation by \( -y'_1(a) \), the second equation by \( y_1(a) \), and add, we get
\[ [y_1(a)y'_2(a) - y_2(a)y'_1(a)]C_2 = -\alpha y'_1(a) + \beta y_1(a). \]

We are guaranteed that this pair of equations has solutions \( C_1, C_2 \) if and only if
\[ y_1(a)y'_2(a) - y_2(a)y'_1(a) \neq 0 \]
in which case
\[
C_1 = \frac{\alpha y'_2(a) - \beta y_2(a)}{y_1(a)y'_2(a) - y_2(a)y'_1(a)} \quad \text{and} \quad C_2 = \frac{-\alpha y'_1(a) + \beta y_1(a)}{y_1(a)y'_2(a) - y_2(a)y'_1(a)}. \]
Since \( a \) was chosen to be any point on \( I \), we conclude that (1) is the general solution of (H) if and only if
\[ y_1(x)y'_2(x) - y_2(x)y'_1(x) \neq 0 \quad \text{for all} \quad x \in I. \]
**DEFINITION 1.** (Wronskian) Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). The function \( W \) defined by

\[
W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)
\]

is called the Wronskian of \( y_1, y_2 \).

We use the notation \( W[y_1, y_2](x) \) to emphasize that the Wronskian is a function of \( x \) that is determined by two solutions \( y_1, y_2 \) of equation (H). When there is no danger of confusion, we’ll shorten the notation to \( W(x) \).

**Remark** Note that \( W \) can be written as a determinant

\[
W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).
\]

**THEOREM 2.** Let \( y = y_1(x) \) and \( y = y_2(x) \) be nontrivial solutions of (H), and let \( W(x) \) be their Wronskian. Exactly one of the following holds:

(i) \( W(x) = 0 \) for all \( x \in I \) and \( y_1 \) is a constant multiple of \( y_2 \) or vice versa.

(ii) \( W(x) \neq 0 \) for all \( x \in I \) and \( y = C_1y_1(x) + C_2y_2(x) \) is the general solution of (H)

**Proof:** Let \( y_1, y_2 \) be nontrivial solutions of (H), and let \( W = y_1y_2' - y_2y_1' \). Since \( y_1 \) is a solution of (H),

\[
y''_1 + py'_1 + qy_1 = 0 \quad \text{and} \quad y'_1 = -py_1' - qy_1.
\]

Similarly for \( y_2 \).

Now

\[
W' = y_1y_2'' - y_2y_1'' = y_1[-py_2' - qy_2] - y_2[-py_1' - qy_1] = -p[y_1y_2' - y_2y_1'] = -pW
\]

and \( W' + pW = 0 \). Therefore, \( W \) is a solution of the special first order linear equation (see Section 2.1)

\[
y' + p(x)y = 0.
\]

As we saw in Section 2.1, \( W \) is either identically 0 on \( I \), or never 0 on \( I \).

We leave it as an exercise (Exercise 23) to show that if \( W \equiv 0 \) on some interval, then either \( y_1 = Cy_2 \) or vice versa. ■

**DEFINITION 2.** (Fundamental Set/Solution Basis) A pair of solutions \( y = y_1(x), y = y_2(x) \) of equation (H) forms a fundamental set of solutions or a solution basis if

\[
W[y_1, y_2](x) \neq 0 \quad \text{for all} \ x \in I.
\]
Linear Dependence; Linear Independence

By Theorem 2, if \( y_1 \) and \( y_2 \) are solutions of equation (H) such that \( W[y_1, y_2] \equiv 0 \), then \( y_1 \) and \( y_2 \) are constant multiples of each other. The question as to whether or not one function is a multiple of another function and the consequences of this are of fundamental importance in differential equations and in linear algebra.

In this sub-section we are dealing with functions in general, not just solutions of the differential equation (H)

**DEFINITION 3. (Linear Dependence; Linear Independence)** Given two functions \( f = f(x) \), \( g = g(x) \) defined on an interval \( I \). The functions \( f \) and \( g \) are linearly dependent on \( I \) if and only if there exist two real numbers \( c_1 \) and \( c_2 \), not both zero, such that
\[
c_1f(x) + c_2g(x) \equiv 0 \quad \text{on } I.
\]
The functions \( f \) and \( g \) are linearly independent on \( I \) if they are not linearly dependent.

Linear dependence can be stated equivalently as: \( f \) and \( g \) are linearly dependent on \( I \) if and only if one of the functions is a constant multiple of the other.

The term Wronskian defined above for two solutions of equation (H) can be extended to any two differentiable functions \( f \) and \( g \). Let \( f = f(x) \) and \( g = g(x) \) be differentiable functions on an interval \( I \). The function \( W[f, g] \) defined by
\[
W[f, g](x) = f(x)g'(x) - g(x)f'(x)
\]
is called the Wronskian of \( f, g \).

There is a connection between linear dependence/independence and Wronskian.

**THEOREM 3.** Let \( f = f(x) \) and \( g = g(x) \) be differentiable functions on an interval \( I \). If \( f \) and \( g \) are linearly dependent on \( I \), then \( W(x) = 0 \) for all \( x \in I \) (\( W \equiv 0 \) on \( I \)).

This theorem can be stated equivalently as: Let \( f = f(x) \) and \( g = g(x) \) be differentiable functions on an interval \( I \). If \( W(x) \neq 0 \) for at least one \( x \in I \), then \( f \) and \( g \) are linearly independent on \( I \).

Going back to differential equations, Theorem 2 can be restated as

**THEOREM 2’** Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of equation (H). Exactly one of the following holds:

(i) \( W(x) = 0 \) for all \( x \in I \); \( y_1 \) and \( y_2 \) are linear dependent.
(ii) \( W(x) \neq 0 \) for all \( x \in I; \) \( y_1 \) and \( y_2 \) are linearly independent and
\[
y = C_1 y_1(x) + C_2 y_2(x)
\]
is the general solution of (H).

The statements “\( y_1(x), y_2(x) \) form a fundamental set of solutions of (H),” “\( \{y_1(x), y_2(x)\} \) is a solution basis,” and “\( y_1(x), y_2(x) \) are linearly independent solutions of (H)” are synonymous.

The results of this section can be captured in one statement

\[
\text{The set } S \text{ of solutions of (H), a subspace of } C^2(I), \text{ has dimension 2, the order of the equation.}
\]

**Exercises 3.2**

Verify that the functions \( y_1 \) and \( y_2 \) are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

1. \( y'' - y' - 6y = 0; \quad y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}. \)
2. \( y'' - 9y = 0; \quad y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}. \)
3. \( y'' + 9y = 0; \quad y_1(x) = \cos 3x, \quad y_2(x) = \sin 3x. \)
4. \( y'' - 4y' + 4y = 0; \quad y_1(x) = e^{2x}, \quad y_2(x) = xe^{2x}. \)
5. \( x^2y'' - x(x + 2)y' + (x + 2)y = 0; \quad y_1(x) = x, \quad y_2(x) = xe^x. \)
6. Given the differential equation \( y'' - 3y' - 4y = 0. \)
   (a) Find two values of \( r \) such that \( y = e^{rx} \) is a solution of the equation.
   (b) Determine a fundamental set of solutions and give the general solution of the equation.
   (c) Find the solution of the equation satisfying the initial conditions \( y(0) = 1, \ y'(0) = 0. \)
7. Given the differential equation \( y'' - \left( \frac{2}{x} \right) y' - \left( \frac{4}{x^2} \right) y = 0. \)
   (a) Find two values of \( r \) such that \( y = x^r \) is a solution of the equation.
   (b) Determine a fundamental set of solutions and give the general solution of the equation.
(c) Find the solution of the equation satisfying the initial conditions \( y(1) = 2, \ y'(1) = -1 \).

(d) Find the solution of the equation satisfying the initial conditions \( y(2) = y'(2) = 0 \).

8. Given the differential equation \((x^2 + 2x - 1)y'' - 2(x + 1)y' + 2y = 0\).

(a) Show that the equation has a linear polynomial and a quadratic polynomial as solutions.

(b) Find two linearly independent solutions of the equation and give the general solution.

Show that the given functions are linearly independent on the interval \( I \) and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

9. \( y_1(x) = e^{3x}, \ y_2(x) = e^{-x}; \ I = (-\infty, \infty) \).

10. \( y_1(x) = e^{-x}, \ y_2(x) = xe^{-x}; \ I = (-\infty, \infty) \).

11. \( y_1(x) = 1, \ y_2(x) = x; \ I = (-\infty, \infty) \).

12. \( y_1(x) = \cos 2x, \ y_2(x) = \sin 2x; \ I = (-\infty, \infty) \).

13. \( y_1(x) = x, \ y_2(x) = x^2; \ I = (0, \infty) \).

14. \( y_1(x) = x, \ y_2(x) = x \ln x; \ I = (0, \infty) \).

15. Let \( y = y_1(x) \) be a solution of (H): \( y'' + p(x)y' + q(x)y = 0 \) where \( p \) and \( q \) are continuous function on an interval \( I \). Let \( \alpha \in I \) and assume that \( y_1(x) \neq 0 \) on \( I \). Set

\[
y_2(x) = y_1(x) \int e^{-\int p(u) \, du} \frac{y_1'(t)}{y_1^2(t)} \, dt.
\]

Show that \( y_2 \) is a solution of (H) and that \( y_1 \) and \( y_2 \) are linearly independent.

Use Exercise 15 to find a fundamental set of solutions of the given equation starting from the given solution \( y_1 \).

16. \( y'' - 6y' + 8y = 0; \ y_1(x) = e^{2x} \).

17. \( y'' - 6y' + 9y = 0; \ y_1(x) = e^{3x} \).

18. \( y'' - 2\alpha y' + \alpha^2 y = 0, \ \alpha \) a constant; \( y_1(x) = e^{\alpha x} \).

19. \( y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0; \ y_1(x) = x \).

20. \( y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0; \ y_1(x) = x^3 \).
21. \( y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0; \quad y_1(x) = x^2. \)

22. \( x^2 y'' - (2\alpha - 1) x y' + \alpha^2 y = 0, \quad \alpha \text{ a constant}; \quad y_1(x) = x^\alpha. \)

23. \( y'' - \frac{1}{x} y' - 4x^2 y = 0; \quad y_1(x) = e^{x^2}. \)

24. \( y'' - \frac{2x - 1}{x} y' + \frac{x - 1}{x} y = 0; \quad y_1(x) = e^x. \)

Problems 25–30 are concerned with the second order linear homogeneous equation

\[ y'' + p(x) y' + q(x) y = 0 \]  

(H)

where \( p \) and \( q \) are continuous functions on an interval \( I \).

25. Let \( y = y(x) \) be a solution of (H). Suppose there is a number \( a \in I \) such that \( y(a) = y'(a) = 0 \). Prove that \( y(x) = 0 \) for all \( x \in I \). That is, prove that \( y \) is the trivial solution. This is equivalent to proving that if \( y = y(x) \) is a nontrivial solution of (H), then \( y(x) \) and \( y'(x) \) can never be 0 simultaneously.

26. Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). Show that if there is a point \( a \in I \) such that either \( y_1(a) = y_2(a) = 0 \) or \( y_1'(a) = y_2'(a) = 0 \), then \( \{y_1, y_2\} \) are linearly dependent. This is equivalent to proving that if \( y_1 \) and \( y_2 \) are independent solutions of (H), then \( y_1 \) and \( y_2 \) can never be 0 simultaneously, and \( y_1' \) and \( y_2' \) can never be zero simultaneously.

27. Suppose that \( y = y_1(x) \) and \( y = y_2(x) \) are solutions of (H). Show that if \( y_1(x) \neq 0 \) on the interval \( (a, b) \subset I \) and \( W[y_1, y_2](x) \equiv 0 \) on \( (a, b) \), then \( y_2(x) = \lambda y_1(x) \) on \( (a, b) \).

28. Suppose that \( y = y_1(x) \) and \( y = y_2(x) \) are linearly independent solutions of (H). Show that if \( y_1(a) = y_1(b) = 0, \ a < b, \ a,b \in I, \) then \( y_2(c) = 0 \) for some \( c \in (a, b) \).

29. Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). Let \( a \in I \) and suppose that

\[ y_1(a) = \alpha, \quad y_1'(a) = \beta \quad \text{and} \quad y_2(a) = \gamma, \quad y_2'(a) = \delta. \]

Under what conditions on \( \alpha, \beta, \gamma, \delta \) will the functions \( y_1 \) and \( y_2 \) be linearly independent on \( I \)?

30. Suppose that the functions \( y_1 \) and \( y_2 \) are linearly independent solutions of (H). Does it follow that \( c_1 y_1 \) and \( c_2 y_2 \) are also linearly independent solutions of (H)? If not, why not.

31. Suppose that the functions \( y_1 \) and \( y_2 \) are linearly independent solutions of (H). Prove that \( y_3 = y_1 + y_2 \) and \( y_4 = y_1 - y_2 \) are also linearly independent solutions of (H). Conversely, prove that if \( y_3 \) and \( y_4 \) are linearly independent solutions of (H), then \( y_1 \) and \( y_2 \) are linearly independent solutions of (H).
32. Suppose that the functions \( y_1 \) and \( y_2 \) are linearly independent solutions of (II).

Under what conditions will the functions \( y_3 = \alpha y_1 + \beta y_2 \) and \( y_4 = \gamma y_1 + \delta y_2 \) be linearly independent solutions of (II)?

### 3.3 Homogenous Equations with Constant Coefficients

We have emphasized that there are no general methods for solving second (or higher) order linear differential equations. However, there are some special cases for which solution methods do exist. In this section we consider such a case, linear equations with constant coefficients. An extension to a class of equations with non-constant coefficients, so-called Euler Equations, is given at the end of the section.

A second order, linear, homogeneous differential equation with constant coefficients is an equation which can be written in the form

\[
y'' + ay' + by = 0
\]

(1)

where \( a \) and \( b \) are real numbers.

You have seen that the function \( y = e^{-ax} \) is a solution of the first-order linear equation

\[
y' + ay = 0,
\]

the equation modeling exponential growth and decay. This suggests that equation (1) may also have an exponential function \( y = e^{rx} \) as a solution.

If \( y = e^{rx} \), then \( y' = re^{rx} \) and \( y'' = r^2 e^{rx} \). Substitution into (1) gives

\[
r^2 e^{rx} + a (re^{rx}) + b (e^{rx}) = e^{rx} \left( r^2 + ar + b \right) = 0.
\]

Since \( e^{rx} \neq 0 \) for all \( x \), we conclude that \( y = e^{rx} \) is a solution of (1) if and only if

\[
r^2 + ar + b = 0. \tag{2}
\]

Thus, if \( r \) is a root of the quadratic equation (2), then \( y = e^{rx} \) is a solution of equation (1); we can find solutions of (1) by finding the roots of the quadratic equation (2).

**DEFINITION 1.** Given the differential equation (1). The corresponding quadratic equation

\[
r^2 + ar + b = 0
\]

is called the characteristic equation of (1); the quadratic polynomial \( r^2 + ar + b \) is called the characteristic polynomial. The roots of the characteristic equation are called the characteristic roots. ■

The nature of the solutions of the differential equation (1) depends on the nature of the roots of its characteristic equation (2). There are three cases to consider:
(1) Equation (2) has two, distinct real roots, \( r_1 = \alpha, \ r_2 = \beta. \)

(2) Equation (2) has only one real root, \( r = \alpha. \)

(3) Equation (2) has complex conjugate roots, \( r_1 = \alpha + i \beta, \ r_2 = \alpha - i \beta, \ \beta \neq 0. \)

**Case I:** The characteristic equation has two, distinct real roots, \( r_1 = \alpha, \ r_2 = \beta. \) In this case,

\[
y_1(x) = e^{\alpha x} \quad \text{and} \quad y_2(x) = e^{\beta x}
\]

are solutions of (1). Since \( \alpha \neq \beta, \) \( y_1 \) and \( y_2 \) are not constant multiples of each other, the pair \( \{e^{\alpha x}, e^{\beta x}\} \) forms a fundamental set of solutions of equation (1) and

\[
y = C_1 e^{\alpha x} + C_2 e^{\beta x}
\]

is the general solution.

**Note:** We can use the Wronskian to verify the independence of \( y_1 \) and \( y_2: \)

\[
W(x) = y_1 y_2' - y_2 y_1' = e^{\alpha x} (\beta e^{\beta x}) - e^{\beta x} (\alpha e^{\alpha x}) = (\alpha - \beta) e^{(\alpha+\beta)x} \neq 0.
\]

**Example 1.** Find the general solution of the differential equation

\[
y'' + 2y' - 8y = 0.
\]

**SOLUTION** The characteristic equation is

\[
r^2 + 2r - 8 = 0
\]

\[
(r + 4)(r - 2) = 0
\]

The characteristic roots are: \( r_1 = -4, \ r_2 = 2. \) The functions \( y_1(x) = e^{-4x}, \ y_2(x) = e^{2x} \) form a fundamental set of solutions of the differential equation and

\[
y = C_1 e^{-4x} + C_2 e^{2x}
\]

is the general solution of the equation. ■

**Case II:** The characteristic equation has only one real root, \( r = \alpha. \) Then

\[
y_1(x) = e^{\alpha x} \quad \text{and} \quad y_2(x) = x e^{\alpha x}
\]

are linearly independent solutions of equation (1) and

\[
y = C_1 e^{\alpha x} + C_2 x e^{\alpha x}
\]

---

1In this case, \( \alpha \) is said to be a *double root* or a root of *multiplicity* 2.
is the general solution.

**Proof:** We know that \( y_1(x) = e^{\alpha x} \) is one solution of the differential equation; we need to find another solution which is independent of \( y_1 \). Since the characteristic equation has only one real root, \( \alpha \), the equation must be

\[
r^2 + ar + b = (r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0
\]

and the differential equation (1) must have the form

\[
y'' - 2\alpha y' + \alpha^2 y = 0. \tag{\star}
\]

Now, \( z = Ce^{\alpha x}, \ C \) any constant, is also a solution of \((\star)\), but \( z \) is not independent of \( y_1 \) since it is simply a multiple of \( y_1 \). We replace \( C \) by a function \( u \) which is to be determined (if possible) so that \( y = ue^{\alpha x} \) is a solution of \((\star))\). Calculating the derivatives of \( y \), we have

\[
y = ue^{\alpha x} \\
y' = \alpha ue^{\alpha x} + u'e^{\alpha x} \\
y'' = \alpha^2 u e^{\alpha x} + 2\alpha u' e^{\alpha x} + u'' e^{\alpha x}
\]

Substitution into \((\star))\) gives

\[
\alpha^2 u e^{\alpha x} + 2\alpha u' e^{\alpha x} + u'' e^{\alpha x} - 2\alpha [\alpha u e^{\alpha x} + u' e^{\alpha x}] + \alpha^2 u e^{\alpha x} = 0.
\]

This reduces to

\[
u'' e^{\alpha x} = 0 \quad \text{which becomes} \quad u'' = 0 \quad \text{since} \quad e^{\alpha x} \neq 0.
\]

So, we have discovered that if \( u \) is to be a function such that \( y = ye^{\alpha x} \) is a solution of \((\star))\), then \( u'' = 0 \). This is the simplest second order linear differential equation with constant coefficients; the general solution is

\[
u = C_1 + C_2 x = C_1 \cdot 1 + C_2 \cdot x, \quad \text{(you saw this in Chapter 1)}
\]

and \( u_1(x) = 1 \) and \( u_2(x) = x \) form a fundamental set of solutions.

Since \( y = ye^{\alpha x} \), we conclude that

\[
y_1(x) = 1 \cdot e^{\alpha x} = e^{\alpha x} \quad \text{and} \quad y_2(x) = xe^{\alpha x}
\]

are solutions of \((\star))\). It’s easy to see that \( y_1 \) and \( y_2 \) form a fundamental set of solutions of \((\star))\); they are not constant multiples of each other. This can also be checked by using the Wronskian:

\[
W(x) = e^{\alpha x} \left[ e^{\alpha x} + \alpha x e^{\alpha x} \right] - \alpha x e^{\alpha x} = e^{2\alpha x} \neq 0.
\]

This is an application of a general method called *variation of parameters*. We will use the method several times in the work that follows.
Finally, the general solution of (*) is
\[ y = C_1 e^{\alpha x} + C_2 x e^{\alpha x} \]

**Example 2.** Find the general solution of the differential equation
\[ y'' - 6y' + 9y = 0. \]

**SOLUTION** The characteristic equation is
\[ r^2 - 6r + 9 = 0 \]
\[ (r - 3)^2 = 0 \]
There is only one characteristic root: \( r_1 = r_2 = 3 \). The functions \( y_1(x) = e^{3x} \), \( y_2(x) = x e^{3x} \) are linearly independent solutions of the differential equation and
\[ y = C_1 e^{3x} + C_2 x e^{3x} \]
is the general solution.

**Case III:** The characteristic equation has complex conjugate roots:
\[ r_1 = \alpha + i \beta, \ r_2 = \alpha - i \beta, \ \beta \neq 0 \]
In this case
\[ y_1(x) = e^{\alpha x} \cos \beta x \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin \beta x \]
are linearly independent solutions of equation (1) and
\[ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x] \]
is the general solution.

**Proof:** It is true that the functions \( z_1(x) = e^{(\alpha+i\beta)x} \) and \( z_2(x) = e^{(\alpha-i\beta)x} \) are linearly independent solutions of (1), but these are complex-valued functions and we want real-valued solutions. The characteristic equation in this case is
\[ r^2 + ar + b = (r - [\alpha + i \beta])(r - [\alpha - i \beta]) = r^2 - 2\alpha r + \alpha^2 + \beta^2 = 0 \]
and the differential equation (1) has the form
\[ y'' - 2\alpha y' + (\alpha^2 + \beta^2) y = 0. \quad (\ast) \]
We'll proceed in a manner similar to Case II. Set \( y = u e^{\alpha x} \) where \( u \) is to be determined (if possible) so that \( y \) is a solution of \((*)\). Calculating the derivatives of \( y \), we have
\[ y = u e^{\alpha x} \]
\[ y' = \alpha u e^{\alpha x} + u' e^{\alpha x} \]
\[ y'' = \alpha^2 u e^{\alpha x} + 2\alpha u' e^{\alpha x} + u'' e^{\alpha x} \]
Substitution into (*) gives
\[ \alpha^2 u e^{\alpha x} + 2\alpha u' e^{\alpha x} + u'' e^{\alpha x} - 2\alpha [\alpha u e^{\alpha x} + u' e^{\alpha x}] + (\alpha^2 + \beta^2) u e^{\alpha x} = 0. \]

This reduces to
\[ u'' e^{\alpha x} + \beta^2 u e^{\alpha x} = 0 \]
which becomes \( u'' + \beta^2 u = 0 \) since \( e^{\alpha x} \neq 0 \).

Now,
\[ u'' + \beta^2 u = 0 \]
is the equation of simple harmonic motion (for example, it models the oscillatory motion of a weight suspended on a spring). The functions \( u_1(x) = \cos \beta x \) and \( u_2(x) = \sin \beta x \) form a fundamental set of solutions. (Verify this.)

Since \( y = u e^{\alpha x} \), we conclude that
\[ y_1(x) = e^{\alpha x} \cos \beta x \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin \beta x \]
are solutions of (*). Since \( y_1 \) and \( y_2 \) are not constant multiples of each other, they form a fundamental set of solutions. This can also be checked by using the Wronskian.

Finally, we conclude that the general solution of equation (1) is:
\[ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]. \]

Example 3. Find the general solution of the differential equation
\[ y'' - 4y' + 13y = 0. \]

SOLUTION The characteristic equation is: \( r^2 - 4r + 13 \). By the quadratic formula, the roots are
\[ r_1, r_2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i. \]

The characteristic roots are the complex numbers: \( r_1 = 2 + 3i, \ r_2 = 2 - 3i \). The functions \( y_1(x) = e^{2x} \cos 3x, \ y_2(x) = e^{2x} \sin 3x \) are linearly independent solutions of the differential equation and
\[ y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x = e^{2x} [C_1 \cos 3x + C_2 \sin 3x] \]
is the general solution.

Recovering a Differential Equation from Solutions

You can also work backwards using the results above. That is, we can determine a second order, linear, homogeneous differential equation with constant coefficients that has given functions \( u \) and \( v \) as solutions. Here are some examples.
Example 4. Find a second order, linear, homogeneous differential equation with constant coefficients that has the functions \( u(x) = e^{2x} \), \( v(x) = e^{-3x} \) as solutions.

*SOLUTION* Since \( e^{2x} \) is a solution, \( 2 \) must be a root of the characteristic equation and \( r - 2 \) must be a factor of the characteristic polynomial. Similarly, \( e^{-3x} \) a solution means that \( -3 \) is a root and \( r - (-3) = r + 3 \) is a factor of the characteristic polynomial. Thus the characteristic equation must be

\[
(r - 2)(r + 3) = 0 \quad \text{which expands to} \quad r^2 + r - 6 = 0.
\]

Therefore, the differential equation is

\[
y'' + y' - 6y = 0. \quad \blacksquare
\]

Example 5. Find a second order, linear, homogeneous differential equation with constant coefficients that has \( y(x) = e^x \cos 2x \) as a solution.

*SOLUTION* Since \( e^x \cos 2x \) is a solution, the characteristic equation must have the complex numbers \( 1 + 2i \) and \( 1 - 2i \) as roots. (Although we didn’t state it explicitly, \( e^x \sin 2x \) must also be a solution.) The characteristic equation must be

\[
(r - [1 + 2i])(r - [1 - 2i]) = 0 \quad \text{which expands to} \quad r^2 - 2r + 5 = 0
\]

and the differential equation is

\[
y'' - 2y' + 5y = 0. \quad \blacksquare
\]

For the fun of it, go back to Exercises 1.2 and re-do Problems 13, 16, 17, 20, 21, 22 using what we know now.

**Second Order Euler Equations**

A second order Euler equation is a linear differential equation having the special form

\[
\frac{d^2y}{dx^2} + \frac{a}{x} \cdot \frac{dy}{dx} + \frac{b}{x^2}y = 0 \quad \text{on} \quad I = (0, \infty) \tag{E}
\]

This equation is often written in the equivalent form

\[
x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = 0.
\]

We have been using specific equations of this form in examples and exercises in Chapter 1 and in Section 3.2. See, for example, Exercises 1.1, Problems 25 - 30, and Exercises 3.2, Problems 19 - 21. Note that you were asked to find solutions of the form \( y = x^r \).

Clearly, an Euler equation does not have constant coefficients. The significance of an Euler equation is that it can be transformed into an equation with constant coefficients by
means of a change of independent variable. The transformed equation can be solved by the methods of this section; solutions of the Euler equation are then obtained by reversing the change of variable.

**THEOREM 1.** The change of independent variable \( z = \ln x \) transforms (E) into the constant coefficient equation

\[
\frac{d^2y}{dz^2} + (a - 1) \frac{dy}{dz} + by = 0.
\]

**PROOF:** Introduce a new independent variable \( z \) by letting \( z = \ln x \). Then, by the chain rule

\[
\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}.
\]

and, by the product rule and chain rule,

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dz} \cdot \frac{1}{x} \right] = \frac{dy}{dz} \cdot \frac{d}{dx} \left( \frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} \left( \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \left[ \frac{d}{dz} \left( \frac{dy}{dz} \right) \cdot \frac{dz}{dx} \right]
\]

Now, substituting these expressions for \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) into (E), we get

\[
-\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} = -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2y}{dz^2}
\]

which simplifies to

\[
\frac{1}{x^2} \cdot \frac{d^2y}{dz^2} + \frac{1}{x^2} \left[ (a - 1) \frac{dy}{dz} \right] + \frac{b}{x^2} y = 0 \quad \text{or} \quad \frac{1}{x^2} \left[ \frac{d^2y}{dz^2} + (a - 1) \frac{dy}{dz} + by \right] = 0
\]

Since \( \frac{1}{x^2} \neq 0 \) the latter equation is equivalent to

\[
\frac{d^2y}{dz^2} + (a - 1) \frac{dy}{dz} + by = 0. \quad \square
\]

**Example 6.** Find the general solution of the Euler equation: \( x^2 y'' - 2xy' - 10y = 0 \).

**SOLUTION:** The change of variable \( z = \ln x \) transforms this equation into

\[
\frac{d^2y}{dz^2} - 3 \frac{dy}{dz} - 10y = 0.
\]

The characteristic equation is \( r^2 - 3r - 10 = (r - 5)(r + 2) = 0 \) so \( r_1 = 5, \ r_2 = -2 \). The general solution of the transformed equation is

\[
y = C_1 e^{5z} + C_2 e^{-2z}
\]

Now, replacing \( z \) by \( \ln x \), get

\[
y = C_1 e^{5 \ln x} + C_2 e^{-2 \ln x} = C_1 e^{\ln x^5} + C_2 e^{\ln x^{-2}} = C_1 x^5 + C_2 x^{-2}
\]

and this is the general solution of the given equation. \( \square \)
Example 7. Find the general solution of the Euler equation: \( x^2 y'' - 3xy' + 4y = 0 \).

SOLUTION: The change of variable \( z = \ln x \) transforms this equation into

\[
\frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 4y = 0.
\]

The characteristic equation is \( r^2 - 4r + 4 = (r - 2)^2 = 0 \) so \( r_1 = r_2 = 2 \). The general solution of the transformed equation is

\[
y = C_1 e^{2z} + C_2 xe^{2z}
\]

Now, replacing \( z \) by \( \ln x \), get

\[
y = C_1 e^{2 \ln x} + C_2 \ln x e^{2 \ln x} = C_1 e^{\ln x^2} + C_2 \ln x e^{\ln x^2} = C_1 x^2 + C_2 x^2 \ln x
\]

and this is the general solution of the given equation. □

Exercises 3.3

Find the general solution of the given differential equation.

1. \( y'' + 2y' - 8y = 0 \).
2. \( y'' - 13y' + 42y = 0 \).
3. \( y'' - 10y' + 25y = 0 \).
4. \( y'' + 2y' + 5y = 0 \).
5. \( y'' + 4y' + 13y = 0 \).
6. \( y'' + 2y' = 0 \).
7. \( 2y'' + 5y' - 3y = 0 \).
8. \( y'' - 9y = 0 \).
9. \( y'' + 9y = 0 \).
10. \( y'' - 2y' + 2y = 0 \).
11. \( y'' - 3y' + \frac{9}{4} y = 0 \).
12. \( y'' + 7y' + 12y = 0 \).
13. \( y'' - y' - 30y = 0 \).
14. \( y'' + 8y' + 16y = 0 \).
15. \( y'' - 6y' + 10y = 0 \)
16. \( y'' - 4y' + 20y = 0 \)

Find the solution of the initial-value problem.
17. \( y'' - 5y' + 6y = 0; \quad y(0) = 1, \quad y'(0) = 1. \)
18. \( y'' + 4y' + 3y = 0; \quad y(0) = 2, \quad y'(0) = -1. \)
19. \( y'' + 2y' + y = 0; \quad y(0) = -3, \quad y'(0) = 1. \)
20. \( y'' + \frac{1}{2}y = 0; \quad y(\pi) = 1, \quad y'(\pi) = -1. \)
21. \( y'' - 2y' + 2y = 0; \quad y(0) = -1, \quad y'(0) = -1. \)
22. \( y'' + 4y' + 4y = 0; \quad y(-1) = 2, \quad y'(-1) = 1. \)

Find a differential equation \( y'' + ay' + by = 0 \) that is satisfied by the given functions.
23. \( y_1(x) = e^{2x}, \quad y_2(x) = e^{-5x}. \)
24. \( y_1(x) = 3e^{3x}, \quad y_2(x) = 2xe^{3x}. \)
25. \( y_1(x) = \cos 2x, \quad y_2(x) = 2\sin 2x. \)
26. \( y_1(x) = e^{-2x}\cos 4x, \quad y_2(x) = e^{-2x}\sin 4x. \)

Find a differential equation \( y'' + ay' + by = 0 \) whose general solution is the given expression.
27. \( y = C_1 e^{3x} + C_2 e^{-4x}. \)
28. \( y = C_1 e^{-x}\cos 3x + C_2 e^{-x}\sin 3x. \)
29. \( y = C_1 e^{x/2} + C_2 xe^{x/2}. \)
30. \( y = C_1 \cos 4x + C_2 \sin 4x. \)
31. Find the solution \( y = y(x) \) of the initial-value problem \( y'' - y' - 2y = 0; \quad y(0) = \alpha, \quad y'(0) = 2. \) Then find \( \alpha \) such that \( y(x) \to 0 \) as \( x \to \infty. \)
32. Find the solution \( y = y(x) \) of the initial-value problem \( 4y'' - y = 0; \quad y(0) = 2, \quad y'(0) = \beta. \) Then find \( \beta \) such that \( y(x) \to 0 \) as \( x \to \infty. \)
33. Given the differential equation \( y'' - (2a - 1)y' + a(a - 1)y = 0. \)
   (a) Determine the values of \( a \) (if any) for which all solutions have limit 0 as \( x \to \infty. \)
   (b) Determine the values of \( a \) (if any) for which all solutions are unbounded as \( x \to \infty. \)
34. Consider \( y'' + ay' + by = 0 \) where \( a \) and \( b \) are constants. Give a condition on \( a \) and \( b \) which will imply that:

(a) (1) has solutions of the form \( y_1 = e^{\alpha x}, \ y_2 = e^{\beta x} \), \( \alpha, \beta \) distinct real numbers.
(b) (1) has solutions of the form \( y_1 = e^{\alpha x}, \ y_2 = xe^{\alpha x} \), \( \alpha \) a real number.
(c) (1) has solutions of the form \( y_1 = e^{\alpha x} \cos \beta x, \ y_2 = e^{\alpha x} \sin \beta x \), \( \alpha, \beta \) real numbers.

**Exercises 35 - 36** are concerned with the long-term behavior of solutions of the differential equation (1): \( y'' + ay' + by = 0 \) where \( a \) and \( b \) are constants.

35. Prove that if \( a \) and \( b \) are both positive, then all solutions have limit 0 as \( x \to \infty \).

36. Prove:

(a) If \( a = 0 \) and \( b > 0 \), then all solutions of the equation are bounded.
(b) If \( a > 0 \) and \( b = 0 \), and \( y = y(x) \) is a solution, then

\[
\lim_{x \to \infty} y(x) = k \quad \text{for some constant } k.
\]

Determine \( k \) for the solution that satisfies the initial conditions \( y(0) = \alpha, \ y'(0) = \beta \).

37. Show that the general solution of the differential equation

\[
y'' - \omega^2 y = 0, \quad \omega \text{ a positive constant},
\]

can be written

\[
y = C_1 \cosh \omega x + C_2 \sinh \omega x.
\]

38. Suppose that the characteristic \( r^2 + ar + b = 0 \) of the homogeneous equation

\[
y'' + ay' + by = 0
\]

has a double root \( \alpha \). Then the differential equation is

\[
y'' - 2\alpha y' + \alpha^2 y = 0
\]

and \( y_1(x) = e^{\alpha x} \) is one solution. Use Problem 15 in Exercises 3.2 to derive the second, independent solution \( y_2 = xe^{\alpha x} \).

39. Calculate the Wronskian of the functions \( y_1 = e^{\alpha x} \cos \beta x, y_2 = e^{\alpha x} \sin \beta x \)

Find the general solution of the Euler equations.

40. \( \frac{y''}{x} - \frac{y'}{x^2} - \frac{8}{x^2} y = 0 \).

41. \( x^2 y'' - 2xy' + 2y = 0 \).
3.4 Nonhomogeneous Equations: Variation of Parameters

In this section we consider the general second order, linear, nonhomogeneous equation

\[ y'' + p(x)y' + q(x)y = f(x) \]  \hspace{1cm} (N)

where \( p, q, f \) are continuous functions on an interval \( I \).

The objectives of this section are to determine the “structure” of the set of solutions of (N).

As we shall see, there is a close connection between equation (N) and the associated homogeneous equation

\[ y'' + p(x)y' + q(x)y = 0. \]  \hspace{1cm} (H)

In this context, equation (H) is called the reduced equation of equation (N).

General Results

**THEOREM 1.** If \( z = z_1(x) \) and \( z = z_2(x) \) are solutions of equation (N), then

\[ y(x) = z_1(x) - z_2(x) \]

is a solution of equation (H).

**PROOF:** Let \( L \) be the linear operator: \( L[y] = y'' + p(x)y' + q(x)y \). Since \( z_1 \) and \( z_2 \) are solutions of (N),

\[ L[z_1(x)] = f(x) \quad \text{and} \quad L[z_2(x)] = f(x). \]

Now, \( L \) is a linear operator, so

\[ L[z_1 - z_2] = L[z_1] - L[z_2] = f(x) - f(x) = 0. \]

Therefore, \( y(x) = z_1(x) - z_2(x) \) is a solution of (H). \( \blacksquare \)

*The difference of any two solutions of the nonhomogeneous equation (N) is a solution of its reduced equation (H).*

Our next theorem gives the “structure” of the set of solutions of (N).
THEOREM 2. Let \( y = y_1(x) \) and \( y = y_2(x) \) be linearly independent solutions of the reduced equation \( (H) \) and let \( z = z(x) \) be a particular solution of \( (N) \). If \( u = u(x) \) is any solution of \( (N) \), then there exist constants \( C_1 \) and \( C_2 \) such that
\[
u(x) = C_1y_1(x) + C_2y_2(x) + z(x).
\]

PROOF: Let \( u = u(x) \) be any solution of \( (N) \) and set \( y(x) = u(x) - z(x) \). By Theorem 1, \( y \) is a solution of the reduced equation \( (H) \). Since \( y_1 \) and \( y_2 \) are linearly independent solutions of \( (H) \), \( \{y_1, y_2\} \) is a fundamental set of solutions, there exists a unique pair of constants \( C_1, C_2 \) such that
\[
y(x) = u(x) - z(x) = C_1y_1(x) + C_2y_2(x)
\]
which implies \( u(x) = C_1y_1(x) + C_2y_2(x) + z(x) \).

According to Theorem 2, if \( y = y_1(x) \) and \( y = y_2(x) \) are linearly independent solutions of the reduced equation \( (H) \) and \( z = z(x) \) is a particular solution of \( (N) \), then
\[
y = C_1y_1(x) + C_2y_2(x) + z(x)
\]
represents the set of all solutions of \( (N) \). That is, (1) is the general solution of \( (N) \).

Another way to look at (1) is: The general solution of \( (N) \) consists of the general solution of the reduced equation \( (H) \) plus a particular solution of \( (N) \):
\[
\overbrace{y}^{\text{general solution of } (N)} = \overbrace{C_1y_1(x) + C_2y_2(x)}^{\text{general solution of } (H)} + \overbrace{z(x)}^{\text{particular solution of } (N)}.
\]
We noted this result for first order linear equations in Section 2.1 (see p. 24).

The next result is sometimes useful in finding particular solutions of nonhomogeneous equations. It is known as the superposition principle.

THEOREM 3. If \( z = z_1(x) \) and \( z = z_2(x) \) are particular solutions of
\[
y'' + p(x)y' + q(x)y = f(x) \quad \text{and} \quad y'' + p(x)y' + q(x)y = g(x),
\]
respectively, then \( z(x) = z_1(x) + z_2(x) \) is a particular solution of
\[
y'' + p(x)y' + q(x)y = f(x) + g(x).
\]

PROOF: Exercise

This result can be extended to nonhomogeneous equations whose right-hand side is the sum of an arbitrary number of functions.

COROLLARY If \( z = z_1(x) \) is a particular solution of
\[
y'' + p(x)y' + q(x)y = f_1(x),
\]
we...
\[ z = z_2(x) \] is a particular solution of
\[ y'' + p(x)y' + q(x)y = f_2(x), \]
and so on

\[ z = z_n(x) \] is a particular solution of
\[ y'' + p(x)y' + q(x)y = f_n(x), \]
then \[ z(x) = z_1(x) + z_2(x) + \cdots + z_n(x) \] is a particular solution of
\[ y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x). \quad \blacksquare \]

The importance of Theorem 3 and its Corollary is that we need only consider non-homogeneous equations in which the function on the right-hand side consists of one term only.

**Variation of Parameters**

By our work above, to find the general solution of (N) we need to find:

(i) a linearly independent pair of solutions \( y_1, y_2 \) of the reduced equation (H), and

(ii) a particular solution \( z \) of (N).

The *method of variation of parameters* uses a pair of linearly independent solutions of the reduced equation to construct a particular solution of (N).

Let \( y_1(x) \) and \( y_2(x) \) be linearly independent solutions of the reduced equation
\[ y'' + p(x)y' + q(x)y = 0. \]
Then
\[ y = C_1y_1(x) + C_2y_2(x) \]
is the general solution. We replace the arbitrary constants \( C_1 \) and \( C_2 \) by functions \( u = u(x) \) and \( v = v(x) \), which are to be determined so that
\[ z(x) = u(x)y_1(x) + v(x)y_2(x) \]
is a particular solution of the nonhomogeneous equation (N). The replacement of the parameters \( C_1 \) and \( C_2 \) by the “variables” \( u \) and \( v \) is the basis for the term “variation of parameters.” Since there are two unknowns \( u \) and \( v \) to be determined we shall impose two conditions on these unknowns. One condition is that \( z \) should solve the differential
differentiation. 

Differentiating \( z \) we get 

\[
z' = u y_1' + y_1 u' + v y_2' + y_2 v'.
\]

For our second condition on \( u \) and \( v \), we set

\[
y_1 u' + y_2 v' = 0. \tag{a}
\]

This condition is chosen because it simplifies the first derivative \( z' \) and because it will lead to a simple pair of equations in the unknowns \( u \) and \( v \). With this condition the equation for \( z' \) becomes

\[
z' = u y_1' + v y_2'. \tag{b}
\]

and

\[
z'' = u y_1'' + y_1' u' + v y_2'' + y_2' v'.
\]

Now substitute \( z, z' \) (given by (b)), and \( z'' \) into the left side of equation (N). This gives

\[
z'' + p z' + q z = (u y_1'' + y_1' u' + v y_2'' + y_2' v') + p(u y_1' + v y_2') + q(u y_1 + v y_2)
\]

\[
= u(y_1'' + p y_1' + q y_1) + v(y_2'' + p y_2' + q y_2) + y_1' u' + y_2' v'.
\]

Since \( y_1 \) and \( y_2 \) are solutions of (H),

\[
y_1'' + p y_1' + q y_1 = 0 \quad \text{and} \quad y_2'' + p y_2' + q y_2 = 0
\]

and so

\[
z'' + p z' + q z = y_1' u' + y_2' v'.
\]

The condition that \( z \) should satisfy (N) is

\[
y_1' u' + y_2' v' = f(x). \tag{c}
\]

Equations (a) and (c) constitute a system of two equations in the two unknowns \( u \) and \( v \):

\[
y_1 u' + y_2 v' = 0 \quad \quad \quad y_1' u' + y_2' v' = f(x)
\]

Obviously this system involves \( u' \) and \( v' \) not \( u \) and \( v \), but if we can solve for \( u' \) and \( v' \), then we can integrate to find \( u \) and \( v \). Solving for \( u' \) and \( v' \), (using, for example, Cramer’s rule) we find that

\[
u' = \frac{-y_2 f}{y_1 y_2' - y_2 y_1'} \quad \text{and} \quad v' = \frac{y_1 f}{y_1 y_2' - y_2 y_1'}
\]
We know that the denominators here are non-zero because the expression
\[ y_1(x)y_2'(x) - y_2(x)y_1'(x) = W(x) \]
is the Wronskian of \( y_1 \) and \( y_2 \), and \( y_1, y_2 \) are linearly independent solutions of the reduced equation.

We can now get \( u \) and \( v \) by integrating:
\[ u = \int \frac{-y_2(x)f(x)}{W(x)} \, dx \quad \text{and} \quad v = \int \frac{y_1(x)f(x)}{W(x)} \, dx. \]

Finally
\[ z(x) = y_1(x) \int \frac{-y_2(x)f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} \, dx \quad (2) \]
is a particular solution of the nonhomogeneous equation (N).

**Remark** This result illustrates why the emphasis is on linear homogeneous equations. To find the general solution of the nonhomogeneous equation (N) we need a fundamental set of solutions of the reduced equation (H) and one particular solution of (N). But, as we have just shown, if we have a fundamental set of solutions of (H), then we can use them to construct a particular solution of (N). Thus, all we really need to solve (N) is a fundamental set of solutions of its reduced equation (H). ■

**WARNING:** The variation of parameters method requires that the nonhomogeneous equation be written in the form (N); that is, the coefficient of \( y'' \) must be 1. ■

**Example 1.** Find a particular solution of the nonhomogeneous equation
\[ y'' - 5y' + 6y = 4e^{2x}. \quad (*) \]

**SOLUTION** The functions \( y_1(x) = e^{2x}, \ y_2(x) = e^{3x} \) are linearly independent solutions of the reduced equation. The Wronskian of \( y_1, y_2 \) is
\[ W(x) = y_1 y_2' - y_2 y_1' = e^{5x}. \]

By the method of variation of parameters, a particular solution of the nonhomogeneous equation is
\[ z(x) = u(x) e^{2x} + v(x) e^{3x} \]
where, from (2),
\[ u(x) = \int \frac{-e^{3x}(4e^{2x})}{e^{5x}} \, dx = \int -4 \, dx = -4x \]
and
\[ v(x) = \int \frac{e^{2x}(4e^{2x})}{e^{5x}} \, dx = \int 4e^{-x} \, dx = -4e^{-x}. \]
(NOTE: Since we are seeking only one function \( u \) and one function \( v \), we have not included arbitrary constants in the integration steps.)

Now
\[
z(x) = -4x e^{2x} - 4e^{-x} e^{3x} = -4x e^{2x} - 4e^{2x}
\]
is a particular solution of the nonhomogeneous equation (\( \ast \)) and
\[
y = C_1 e^{2x} + C_2 e^{3x} - 4x e^{2x} = C_1 e^{2x} + C_2 e^{3x} - 4x e^{2x}
\]
is the general solution (we “absorbed” \(-4e^{2x}\) in the \( C_1 e^{2x} \) term). As you can check
\[
z = -4x e^{2x}
\]
is a particular solution of the nonhomogeneous equation. \( \blacksquare \)

**Example 2.** Find a particular solution of the nonhomogeneous equation
\[
y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 2x^3
\]
given that \( y_1(x) = x \) and \( y_2(x) = x^2 \) are linearly independent solutions of the corresponding reduced equation. Also give the general solution of the nonhomogeneous equation.

**SOLUTION** The Wronskian of \( y_1, y_2 \) is \( W(x) = y_1 y_2' - y_2 y_1' = x(2x) - x^2(1) = x^2 \). By the method of variation of parameters, a particular solution of the nonhomogeneous equation is
\[
z(x) = u(x) x + v(x) x^2
\]
where, from (2),
\[
u(x) = \int \frac{y_1(x) f(x)}{W(x)} \, dx = \int \frac{x(2x^3)}{x^2} \, dx = \int 2 x^2 \, dx = \frac{2}{3} x^3
\]
and
\[
u(x) = \int \frac{-y_2(x) f(x)}{W(x)} \, dx = \int \frac{-x^2(2x^3)}{x^2} \, dx = \int -2 x^3 \, dx = -\frac{1}{2} x^4
\]
(Note: Since we are seeking only one function \( u \) and one function \( v \) we have not included arbitrary constants in the integration steps.)

Now
\[
z(x) = -\frac{1}{2} x^4 \cdot x + \frac{2}{3} x^3 \cdot x^2 = \frac{1}{6} x^5.
\]
is a particular solution of the nonhomogeneous equation (\( \ast \)) and
\[
y = C_1 x + C_2 x^2 + \frac{1}{6} x^5.
\]
is the general solution. \( \blacksquare \)
Exercises 3.4

Verify that the given functions $y_1$ and $y_2$ form a fundamental set of solutions of the reduced equation of the given nonhomogeneous equation, then find a particular solution of the nonhomogeneous equation and give the general solution of the equation. **Remember:** The variation of parameters method requires that the equation be written in the form (N).

1. $y'' - \frac{2}{x^2} y = 3 - x^{-2}; \quad y_1(x) = x^2, \; y_2(x) = x^{-1}$.

2. $y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{2}{x}; \quad y_1(x) = x, \; y_2(x) = x \ln x$.

3. $xy'' - y' - 4x^3 y = 8x^5; \quad y_1(x) = e^{x^2}, \; y_2(x) = e^{-x^2}$.

4. $y'' - \frac{1 + x}{x} y' + \frac{1}{x} y = xe^{2x}; \quad y_1(x) = 1 + x, \; y_2(x) = e^x$.

5. $(x - 1)y'' - xy' + y = (x - 1)^2; \quad y_1(x) = x, \; y_2(x) = e^x$.

6. $xy'' + (2 - 2x)y' + (x - 2)y = e^{2x}; \quad y_1(x) = e^x, \; y_2(x) = \frac{e^x}{x}$.

7. $x^2 y'' - 2xy' + (x^2 + 2)y = x^3 \cos x; \quad y_1(x) = x \cos x, \; y_2(x) = x \sin x$.

8. $x^2 y'' - 2xy' + (x^2 - 2)y = 3x^4; \quad y_1(x) = xe^x, \; y_2(x) = xe^{-x}$.

Find the general solution of the given nonhomogeneous differential equation.

9. $y'' - y' - 2y = 2e^{-x}$.

10. $y'' - 2y' - 3y = 3e^{2x}$.

11. $y'' - 4y' + 4y = 4e^{2x}$.

12. $y'' + y = \tan x$.

13. $y'' + 4y = \sec 2x$.

14. $y'' - 2y' + y = xe^x$.

15. $y'' - 2y' + y = e^x \cos x$.

16. $y'' - 4y' + 4y = \frac{1}{3} x^{-1} e^{2x}$.

17. $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$.

18. $y'' + 2y' + y = e^{-x} \ln x$.

19. $y'' + 9y = 9 \sec^2 3x$.

20. $x^2 y'' - 4xy' + 6y = x^{5/2}$.
21. \( x^2y'' - 2xy' + 2y = x^2 \ln x \).

22. \( x^2y'' - xy' + y = 4x \ln x \).

23. The function \( y_1(x) = x \) is a solution of \( (x^2 + 1)y'' - 2xy' + 2y = 0 \). Find the general solution of the differential equation
\[
(x^2 + 1)y'' - 2xy' + 2y = (x^2 + 1)^2.
\]
HINT: Use Exercise 15, Section 3.2, to construct a \( y_2 \).

24. The function \( y_1(x) = e^x \) is a solution of \( xy'' - (2x + 2)y' + (x + 2)y = 0 \). Find the general solution of the differential equation
\[
xy'' - (2x + 2)y' + (x + 2)y = 6x^3 e^x.
\]

25. The functions \( y_1(x) = x^2 + x \ln x \), \( y_2(x) = x + x^2 \) and \( y_3(x) = x^2 \) are solutions of a second order linear nonhomogeneous equation. What is the general solution of the equation?

26. The functions \( y_1(x) = x - 2x^3 \), \( y_2(x) = xe^x + x - 2x^3 \) and \( y_3(x) = -2x^3 \) are solutions of a second order linear nonhomogeneous equation. What is the general solution of the equation?

27. Prove Theorem 3.

3.5 Nonhomogeneous Equations: Undetermined Coefficients

Solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. We have seen one method for finding a particular solution, the method of variation of parameters. In this section we present another method, the method of undetermined coefficients.

Remark: Limitations of the method. In contrast to variation of parameters, which can be applied to any nonhomogeneous equation, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form
\[
y'' + ay' + by = f(x)
\]
where \( a \) and \( b \) are constants and the nonhomogeneous term \( f \) is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.

To motivate the method of undetermined coefficients, consider the linear operator \( L \) on the left side of (1):
\[
L[y] = y'' + ay' + by.
\]
If we calculate $L$ for an exponential function $z = Ae^{rx}$, $A$ a constant, we have

$$z = Ae^{rx}, \quad z' = Ae^{rx}, \quad z'' = Ar^2e^{rx}$$

and

$$L[y] = y'' + ay' + by = Ar^2e^{rx} + a(Ae^{rx}) + b(Ae^{rx}) = (Ar^2 + aAr + bA)e^{rx}$$

$$= Ke^{rx} \quad \text{where } K = Ar^2 + aAr + bA.$$ 

That is, the operator $L$ transforms $Ae^{rx}$ into a constant multiple of $e^{rx}$. We can use this result to determine a particular solution of a nonhomogeneous equation of the form

$$y'' + ay' + by = ce^{rx}.$$ 

Here is a specific example.

**Example 1.** Find a particular solution of the nonhomogeneous equation

$$y'' - 2y' + 5y = 6e^{3x}.$$ 

**SOLUTION** As we saw above, if we “apply” $L[y] = y'' - 2y' + 5y$ to $z(x) = Ae^{3x}$ we will get an expression of the form $Ke^{3x}$. We want to determine $A$ so that $K = 6$. The constant $A$ is called an undetermined coefficient. We have

$$z = Ae^{3x}, \quad z' = 3Ae^{3x}, \quad z'' = 9Ae^{3x}.$$ 

Substituting $z$ and its derivatives into the left side of the differential equation, we get

$$9Ae^{3x} - 2(3Ae^{3x}) + 5(Ae^{3x}) = (9A - 6A + 5A)e^{3x} = 8Ae^{3x}.$$ 

We want

$$z'' - 2z' + 5z = 6e^{3x},$$

so we set

$$8Ae^{3x} = 6e^{3x} \quad \text{which gives } 8A = 6 \quad \text{and } A = \frac{3}{4}.$$ 

Thus, $z(x) = \frac{3}{4}e^{3x}$ is a particular solution of $y'' - 2y' + 5y = 6e^{3x}$. (Verify this.)

You can also verify that

$$y = C_1e^{x}\cos 2x + C_2e^{x}\sin 2x + \frac{3}{4}e^{3x}$$

is the general solution of the equation. ■

If we set $z(x) = A\cos \beta x$ and calculate $z'$ and $z''$, we get

$$z = A\cos \beta x, \quad z' = -\beta A\sin \beta x, \quad z'' = -\beta^2 A\cos \beta x.$$
Therefore, \( L[y] = y'' + ay' + by \) applied to \( z \) gives

\[
z'' + az' + bz = -\beta^2 A \cos \beta x + a(-\beta A \sin \beta x) + b(A \cos \beta x)
= (-\beta^2 A + bA) \cos \beta x + (-a \beta A) \sin \beta x.
\]

That is, \( L \) “transforms” \( z = A \cos \beta x \) into an expression of the form

\[K \cos \beta x + M \sin \beta x\]

where \( K \) and \( M \) are constants which depend on \( a, b, \beta \) and \( A \). We will get exactly the same type of result if we apply \( L \) to \( z = B \sin \beta x \). Combining these two results, it follows that \( L[y] = y'' + ay' + by \) applied to

\[z = A \cos \beta x + B \sin \beta x\]

will produce the expression

\[K \cos \beta x + M \sin \beta x\]

where \( K \) and \( M \) are constants which depend on \( a, b, \beta, A, \) and \( B \).

Now suppose we have a nonhomogeneous equation of the form

\[y'' + ay' + by = c \cos \beta x \quad \text{or} \quad y'' + ay' + by = d \sin \beta x,\]

or even

\[y'' + ay' + by = c \cos \beta x + d \sin \beta x.\]

Then we will look for a solution of the form \( z(x) = A \cos \beta x + B \sin \beta x \).

**Example 2.** Find the general solution of

\[y'' - 3y' + 2y = 4 \cos 2x + \sin 2x. \quad (*)\]

**SOLUTION:** According to Theorem 2, Section 3.4, we need the general solution of the reduced equation

\[y'' - 3y' + 2y = 0\]

and a particular solution of \((*)\).

As you can check, \( y = C_1 e^x + C_2 e^{2x} \) is the general solution reduced equation.

From our observations above, we can try to find a particular solution \( z \) of the nonhomogeneous equation by setting \( z = A \cos 2x + B \sin 2x \), where \( A \) and \( B \) are coefficients to be determined. We calculate \( z' \) and \( z'' \):

\[z' = -2A \sin 2x + 2B \cos 2x, \quad z'' = -4A \cos 2x - 4B \sin 2x\]
and substitute into (*):

\[-4A \cos 2x - 4B \sin 2x - 3(-2A \sin 2x + 2B \cos 2x) + 2(A \cos 2x + B \sin 2x) = 4 \cos 2x + \sin 2x \]
\[(-2A - 6B) \cos 2x + (6A - 2B) \sin 2x = 4 \cos 2x + \sin 2x. \]

It now follows that

\[-2A - 6B = 4 \]
\[6A - 2B = 1 \]

The solution of this pair of equations is: \( A = -\frac{1}{20}, \ B = -\frac{13}{20} \). Thus,

\[ z = -\frac{1}{20} \cos 2x - \frac{13}{20} \sin 2x \]

is a particular solution of (*) and

\[ y = C_1 e^x + C_2 e^{2x} - \frac{1}{20} \cos 2x - \frac{13}{20} \sin 2x \]

is the general solution of (*). \( \blacksquare \)

Continuing with these ideas, if \( L[y] = y'' + ay' + by \) is applied to

\[ z = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x, \]

then the result will have the form

\[ Ke^{\alpha x} \cos \beta x + Ke^{\alpha x} \sin \beta x \]

where \( K \) and \( M \) are constants which depend on \( a, b, \alpha, \beta, A, B \). Therefore, we expect that a nonhomogeneous equation of the form

\[ y'' + ay' + by = ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x \]

will have a particular solution of the form \( z = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x \).

The following table summarizes our discussion to this point.

<table>
<thead>
<tr>
<th>If ( f(x) = )</th>
<th>try ( z(x) = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ce^{rx} )</td>
<td>( Ae^{rx} )</td>
</tr>
<tr>
<td>( c \cos \beta x + d \sin \beta x )</td>
<td>( z(x) = A \cos \beta x + B \sin \beta x )</td>
</tr>
<tr>
<td>( ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x )</td>
<td>( z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x )</td>
</tr>
</tbody>
</table>

Note: The first line includes the case \( r = 0 \); if \( f(x) = ce^{0x} = c \), then \( z = Ae^{0x} = A \).
Unfortunately, the situation is not quite as simple as it appears; there is a difficulty.

**Example 3.** Find a particular solution of the nonhomogeneous equation

\[ y'' - 5y' + 6y = 4e^{2x}. \]  

**SOLUTION** According to the table, we should set \( z(x) = Ae^{2x} \). Calculating the derivatives of \( z \), we have

\[ z = Ae^{2x}, \quad z' = 2Ae^{2x}, \quad z'' = 4Ae^{2x}. \]

Substituting \( z \) and its derivatives into the left side of \((*)\), we get

\[ z'' - 5z' + 6z = 4Ae^{2x} - 5(2Ae^{2x}) + 6(Ae^{2x}) = 0Ae^{2x}. \]

Clearly the equation

\[ 0Ae^{2x} = 4e^{2x} \quad \text{which is equivalent to} \quad 0A = 4 \]

does not have a solution. Therefore equation \((*)\) does not have a solution of the form \( z = Ae^{2x} \).

The problem here is \( z = Ae^{2x} \) is a solution of the reduced equation

\[ y'' - 5y' + 6y = 0. \]

(The characteristic equation is \( r^2 - 5r + 6 = 0 \); the roots are \( r = 2, 3 \); and \( y_1 = e^{2x}, \ y_2 = e^{3x} \) are linearly independent solutions.)

In Example 1 of the preceding section we saw that \( z(x) = -4xe^{2x} \) is a particular solution of \((*)\). So, in the context here, since our trial solution \( z = Ae^{2x} \) solves the reduced equation, we’ll try \( z = Axe^{2x} \). The derivatives of this \( z \) are:

\[ z = Axe^{2x}, \quad z' = 2Axe^{2x} + Ae^{2x}, \quad z'' = 4Axe^{2x} + 4Ae^{2x}. \]

Substituting into the left side of \((*)\), we get

\[ z'' - 5z' + 6z = 4Axe^{2x} + 4Ae^{2x} - 5(2Axe^{2x} + Ae^{2x}) + 6(Axe^{2x}) \]

\[ = -Ae^{2x}. \]

Setting \( z'' - 5z' + 6z = 4e^{2x} \) gives

\[-Ae^{2x} = 4e^{2x} \quad \text{which implies} \quad A = -4. \]

Thus, \( z(x) = -4xe^{2x} \) is a particular solution of \((*)\) (as we already know).  

We learn from this example that we have to make an adjustment if our trial solution \( z \) (from the table) satisfies the reduced equation. Here’s another example.
**Example 4.** Find a particular solution of
\[ y'' + 6y' + 9y = 5e^{-3x}. \] (**) 

**SOLUTION** The reduced equation, \( y'' + 6y' + 9y = 0 \) has characteristic equation
\[ r^2 + 6r + 9 = (r + 3)^2 = 0. \]

Thus, \( r = -3 \) is a double root and \( y_1(x) = e^{-3x}, \ y_2(x) = xe^{-3x} \) form a fundamental set of solutions.

According to our table, to find a particular solution of (**) we should try \( z = Ae^{-3x}. \) But this won’t work, \( z \) is a solution of the reduced equation. Based on the result of the preceding example, we should try \( z = Axe^{-3x}, \) but this won’t work either; \( z = Axe^{-3x} \) is also a solution of the reduced equation. So we’ll try \( z = Ax^2e^{-3x}. \) You can verify that
\[ z(x) = \frac{5}{2} x^2 e^{-3x} \]
is a particular solution of (**).

The general solution of (**) is: \( y = C_1 e^{-3x} + C_2 xe^{-3x} + \frac{5}{2} x^2 e^{-3x}. \) ■

Based on these examples we amend our table to read:

**Table 1**

A particular solution of \( y'' + ay' + by = f(x) \)

<table>
<thead>
<tr>
<th>If ( f(x) = )</th>
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</tr>
</tbody>
</table>
| \( ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x \) | \( z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x \)

*Note: If \( z \) satisfies the reduced equation, try \( xz; \) if \( xz \) also satisfies the reduced equation, then \( x^2z \) will give a particular solution.

**Remark** In practice it is a good idea to solve the homogeneous equation before selecting the trial solution \( z \) of the nonhomogeneous equation. That way you will not waste your time selecting a \( z \) that satisfies the reduced equation. ■

So far we have only considered the nonhomogeneous differential equation (1) in cases where the nonhomogeneous term \( f \) is a constant multiple of one of the functions \( e^{rx}, \cos \beta x, \)
\[ f(x) = p(x)e^{\alpha x} \]
\[ f(x) = p(x)\cos \beta x, \text{ or } p(x)\sin \beta x, \]
\[ f(x) = p(x)e^{\alpha x}\cos \beta x, \text{ or } p(x)e^{\alpha x}\sin \beta x \]

where \( p \) is a polynomial, or where \( f \) is a sum of such functions. This follows from the fact that the expression \( y'' + ay' + by \) applied to
\[ z = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)e^{\alpha x} \]
will result in an expression of the form \( P(x)e^{\alpha x} \) where \( P \) is a polynomial of degree \( n \) (or less); \( y'' + ay' + by \) applied to
\[ z = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)\cos \beta x \]
will result in an expression of the form \( P(x)\cos \beta x + Q(x)\sin \beta x \) where \( P \) and \( Q \) are polynomials of degree \( n \) (or less); and so on.

**Undetermined Coefficients, General Case:** The general version of the method of undetermined coefficients can be summarized as follows:

1. If \( f(x) = p(x)e^{\alpha x} \) where \( p \) is a polynomial of degree \( n \), then
   \[ z(x) = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)e^{\alpha x}. \]
2. If \( f(x) = p_1(x)\cos \beta x + p_2(x)\sin \beta x \) where \( p_1 \) and \( p_2 \) are polynomials of degrees \( k \) and \( m \), respectively, then
   \[ z(x) = (A_0 + A_1x + \cdots + A_nx^n)\cos \beta x + (B_0 + B_1x + \cdots + B_nx^n)\sin \beta x \]
   where \( n = \max \{k, m\} \).
3. If \( f(x) = p_1(x)e^{\alpha x}\cos \beta x + p_2(x)e^{\alpha x}\sin \beta x \) where \( p_1 \) and \( p_2 \) are polynomials of degrees \( k \) and \( m \), respectively, then
   \[ z(x) = (A_0 + A_1x + \cdots + A_nx^n)e^{\alpha x}\cos \beta x + (B_0 + B_1x + \cdots + B_nx^n)e^{\alpha x}\sin \beta x \]
   where \( n = \max \{k, m\} \).

Note: If any term in \( z \) satisfies the reduced equation \( y'' + ay' + by = 0 \), then use \( xz \) as the trial solution; if any term in \( xz \) satisfies the reduced equation, then \( x^2z \) will give a particular solution.

Here are some examples.
Example 5. Find a particular solution of

\[ y'' + 4y = (3 + 2x)e^{-2x}. \tag{*} \]

**SOLUTION** The functions \( y_1(x) = \cos 2x, \) \( y_2(x) = \sin 2x \) form a fundamental set of solutions of the reduced equation \( y'' + 4y = 0. \)

A particular solution of \((*)\) will have the form \( z = (A + Bx)e^{-2x} \) where \( A \) and \( B \) are to be determined. The derivatives of \( z \) are:

\[ z = (A + Bx)e^{-2x}, \quad z' = -2(A + Bx)e^{-2x} + Be^{-2x}, \quad z'' = 4(A + Bx)e^{-2x} - 4Be^{-2x}. \]

Substituting \( z \) and its derivatives into the left side of \((*)\), we get

\[ z'' + 4z = 4(A + Bx)e^{-2x} - 4Be^{-2x} + 4(A + Bx)e^{-2x} = [(8A - 4B) + 8Bx]e^{-2x}. \]

Thus \( z \) is a solution of \((*)\) if

\[ [(8A - 4B) + 8Bx]e^{-2x} = (3 + 2x)e^{-2x} \]

which implies \( 8A - 4B = 3 \) and \( 8B = 2 \).

The solution of this pair of equations is \( A = \frac{1}{2}, \) \( B = \frac{1}{4} \), and

\[ z(x) = \left(\frac{1}{2} + \frac{1}{4} x\right)e^{-2x} \]

is a particular solution of \((*)\).

Example 6. Give the form of the general solution of each of the following nonhomogeneous equations:

(a) \( y'' - 3y' + 2y = (1 + 2x - 4x^2)e^{2x}. \)

(b) \( y'' + 4y' + 4y = (3 - 5x)e^{-2x}. \)

**SOLUTION** (a) The reduced equation is \( y'' - 3y' + 2y = 0. \) The characteristic equation is

\[ r^2 - 3r + 2 = (r - 1)(r - 2) = 0. \]

Thus, \( y_1(x) = e^x, \) \( y_2(x) = e^{2x} \) forms a fundamental set of solutions of the reduced equation.

According to the summary above, a particular solution \( z \) should have the form

\[ z = (A_0 + A_1x + A_2x^2)e^{2x} \]

but \( A_0e^{2x} \) satisfies the reduced equation. Therefore we need to multiply the trial solution by \( x \) and try

\[ z = (A_0x + A_1x^2 + A_2x^3)e^{2x}. \]
Since none of the terms in this \( z \) satisfies the reduced equation, this is the form of a particular solution.

The general solution of the equation will have the form
\[
y = C_1e^x + C_2e^{2x} + (A_0x + A_1x^2 + A_2x^3)e^{2x}
\]
where \( A_0, A_1, A_2 \) are constants which are to be determined.

(b) The reduced equation is \( y'' + 4y' + 4y = 0 \). The characteristic equation is
\[
r^2 + 4r + 4 = (r + 2)^2 = 0.
\]
Thus, \( y_1(x) = e^{-2x}, \ y_2(x) = xe^{-2x} \) forms a fundamental set of solutions of the reduced equation.

According to the summary above, a particular solution \( z \) should have the form
\[
z = (A_0 + A_1x)e^{-2x}
\]
but \( A_0e^{-2x} \) and \( A_1xe^{-2x} \) satisfy the reduced equation. Therefore we need to multiply the trial solution by \( x \) and try
\[
z = (A_0x + A_1x^2)e^{-2x}.
\]
But \( A_0xe^{-2x} \) also satisfies the reduced equation so we need to multiply the initial \( z \) by \( x^2 \). Since none of the terms in
\[
z = (A_0x^2 + a_1x^3)e^{-2x}
\]
satisfies the reduced equation, this is the form of a particular solution.

The general solution of the equation will have the form
\[
y = C_1e^{-2x} + C_2xe^{-2x} + (A_0x^2 + A_1x^3)e^{-2x}
\]
where \( A_0, A_1 \) are constants which are to be determined.

**Summary** The method of variation of parameters can be applied to any linear nonhomogeneous equations but it has the limitation of requiring a fundamental set of solutions of the reduced equation.

The method of undetermined coefficients is limited to linear nonhomogeneous equations with constant coefficients and with restrictions on the nonhomogeneous term \( f \).

In cases where both methods are applicable, the method of undetermined coefficients is usually simpler and, hence, the preferable method.
Exercises 3.5

Find the general solution.

1. \( y'' - 2y' - 3y = 3e^{2x} \).
2. \( y'' + 2y' + 2y = 10e^x \).
3. \( y'' + 6y' + 9y = 9e^{3x} \).
4. \( y'' + 6y' + 9y = e^{-3x} \).
5. \( y'' + 2y' = 4 \sin 2x \).
6. \( y'' + y = 3 \sin 2x + x \cos 2x \).
7. \( y'' - 6y' + 9y = 6e^{-3x} \).
8. \( y'' + 5y' + 6y = 3x + 4 \).
9. \( y'' + 6y' + 8y = 3e^{-2x} \).
10. \( y'' + 2y' + y = xe^{-x} \).
11. \( y'' - 2y' + 5y = e^{-x} \sin 2x \).
12. \( y'' + 2y' + 5y = e^{2x} \cos x \).

Find the solution of the given initial-value problem.

13. \( y'' + y' - 2y = 2x; \quad y(0) = 0, \quad y'(0) = 1 \).
14. \( y'' - y' - 2y = \sin 2x; \quad y(0) = 1, \quad y'(0) = -1 \).
15. \( y'' + 4y = x^2 + 3e^x; \quad y(0) = 0, \quad y'(0) = 2 \).
16. \( y'' - 2y' + y = xe^x + 4; \quad y(0) = 1, \quad y'(0) = 1 \).

Determine a suitable form for a particular solution \( z = z(x) \) of the given equation.

17. \( y'' - 2y' - 3y = 6 - 3xe^{-x} + 4 \cos 3x \).
18. \( y'' + 2y' = 2x + x^2e^{-3x} + \sin 2x \).
19. \( y'' + y = x^2 - 1 + 3 \cos x - 2 \sin x \).
20. \( y'' - 5y' + 6y = 2e^{2x} \cos x - 3xe^{3x} + 5 \).
21. \( y'' - 4y' + 4y = 2xe^{2x} + x^2 - 1 + 2x \cos 2x \).
22. \( y'' + 5y' + 6y = 2e^{2x} \cos x - 3xe^{3x} + 5e^{-2x} \).
23. \( y'' + 2y' + 2y = 4e^{-x} + 2e^{-x} \cos x + 9. \)

24. \( y'' + 2y' + 5y = 4e^{-x} \sin 2x + 2e^{-x} \cos x. \)

Find the general solution of the given differential equation.

25. \( y'' - 4y' + 4y = 2 \sin x + 3x^{-1}e^{2x}. \)

26. \( y'' - 2y' + y = \frac{e^x}{x^2 + 1} + 2e^{2x}. \)

27. \( y'' + 9y = 3 \cos x - 9 \sec^2 3x. \)

28. \( y'' + 4y = 5e^{4x} + 3 - \sec^2 2x. \)

Exercises 29 and 30 are concerned with the differential equation

\[
y'' + ay' + by = f(x)
\]

where \( a \) and \( b \) are nonnegative constants.

29. Suppose that \( a, b > 0 \). Show that if \( y_1(x) \) and \( y_2(x) \) are solutions of the equation, then \( y_1(x) - y_2(x) \to 0 \) as \( x \to \infty \). What happens if \( a = 0 \) and \( b > 0 \)?

30. If \( f(x) = c, c \) a constant, show that every solution \( y(x) \) of the equation has the property \( y(x) \to c/b \) as \( x \to \infty \). What happens if \( b = 0 \)? What happens if \( a = b = 0 \)?

### 3.6 Vibrating Mechanical Systems

**I. Undamped Free Vibrations (Simple Harmonic Motion).** A spring of length \( l_0 \) units is suspended from a support. When an object of mass \( m \) is attached to the spring, the spring stretches to a length \( l_1 \) units. If the object is then pulled down (or pushed up) an additional \( y_0 \) units at time \( t = 0 \) and then released, what is the resulting motion of the object? That is, what is the position \( y(t) \) of the object at time \( t > 0 \)? Assume that time is measured in seconds.

We begin by analyzing the forces acting on the object at time \( t > 0 \). First, there is the weight of the object (gravity): \( F_1 = mg. \)

This is a downward force. We choose our coordinate system so that the positive direction is down. Next, there is the restoring force of the spring. By Hooke’s Law, this force is proportional to the total displacement \( l_1 + y(t) \) and acts in the direction opposite to the displacement:

\[
F_2 = -k[l_1 + y(t)] \quad \text{with } k > 0.
\]
The constant of proportionality $k$ is called the *spring constant*. If we assume that the spring is frictionless and that there is no resistance due to the surrounding medium (for example, air resistance), then these are the only forces acting on the object. Under these conditions, the total force is

$$F = F_1 + F_2 = mg - k[l_1 + y(t)] = (mg - kl_1) - ky(t).$$

Before the object was displaced, the system was in equilibrium, so the force of gravity, $mg$ plus the force of the spring, $-kl_1$, must have been 0:

$$mg - kl_1 = 0.$$  

Therefore, the total force $F$ reduces to

$$F = -ky(t).$$

By Newton’s Second Law of Motion, $F = ma$ (force = mass $\times$ acceleration), we have

$$ma = -ky(t) \quad \text{and} \quad a = -\frac{k}{m}y(t).$$

Therefore, at any time $t$ we have

$$a = y''(t) = -\frac{k}{m}y(t) \quad \text{or} \quad y''(t) + \frac{k}{m}y(t) = 0.$$  

When the acceleration is a constant negative multiple of the displacement, the object is said to be in *simple harmonic motion*.

Since $k/m > 0$, we can set $\omega = \sqrt{k/m}$ and write this equation as

$$y''(t) + \omega^2 y(t) = 0,$$  

a second order, linear homogeneous equation with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0$$

and the characteristic roots are $\pm \omega i$. The general solution of (1) is

$$y = C_1 \cos \omega t + C_2 \sin \omega t.$$  

In the Exercises you are asked to show that the general solution can be written as

$$y = A \sin (\omega t + \phi_0),$$  

where $A$ and $\phi_0$ are constants with $A > 0$ and $\phi_0 \in [0, 2\pi)$. For our purposes here, this is the preferred form. The motion is periodic with period $T$ given by

$$T = \frac{2\pi}{\omega}.$$  

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a complete oscillation takes $2\pi/\omega$ seconds. The number $\omega$ is called the \textit{natural frequency} of the system. Since $\sin(\omega t + \phi_0)$ oscillates between $-1$ and $1$,

$$y(t) = A\sin(\omega t + \phi_0)$$

oscillates between $-A$ and $A$. The number $A$ is called the \textit{amplitude} of the motion. The number $\phi_0$ is called the \textit{phase constant} or the \textit{phase shift}.

The figure gives a typical graph of (2).

II. Damped Free Vibrations. If the spring is not frictionless and/or if the surrounding medium resists the motion of the object (for example, air resistance), then the resistance tends to dampen the oscillations. Experiments show that such a resistant force $R$ is approximately proportional to the velocity $v = y'$ and acts in a direction opposite to the motion:

$$R = -cy' \quad \text{with} \ c > 0.$$ 

Taking this force into account, the force equation reads

$$F = -ky(t) - cy'(t).$$

Newton’s Second Law $F = ma = my''$ then gives

$$my''(t) = -ky(t) - cy'(t)$$

which can be written as

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \quad (c, \ k, \ m \ \text{all constant}) \quad (3)$$

This is the equation of motion in the presence of a \textit{damping factor}.

The characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$ 

There are three cases to consider:

$$c^2 - 4km < 0, \quad c^2 - 4km > 0, \quad c^2 - 4km = 0.$$
Case 1: \( c^2 - 4km < 0 \). In this case the characteristic equation has complex roots:

\[
    r_1 = -\frac{c}{2m} + i\omega, \quad r_2 = -\frac{c}{2m} - i\omega \quad \text{where} \quad \omega = \frac{\sqrt{4km - c^2}}{2m}.
\]

The general solution is

\[
    y = e^{\left(-\frac{c}{2m}\right)t} \left(C_1 \cos \omega t + C_2 \sin \omega t\right)
\]

which can also be written as

\[
    y(t) = A e^{\left(-\frac{c}{2m}\right)t} \sin (\omega t + \phi_0) \quad (4)
\]

where, as before, \( A \) and \( \phi_0 \) are constants, \( A > 0, \phi_0 \in [0, 2\pi) \). This is called the underdamped case. The motion is similar to simple harmonic motion except that the damping factor \( e^{\left(-\frac{c}{2m}\right)t} \) causes \( y(t) \to 0 \) as \( t \to \infty \).

The oscillations continue indefinitely with constant frequency \( f = \omega/2\pi \) but diminishing amplitude \( Ae^{\left(-\frac{c}{2m}\right)t} \).

The figure below illustrates this motion.

![Figure 2](image)

Case 2: \( c^2 - 4km > 0 \). In this case the characteristic equation has two distinct real roots:

\[
    r_1 = -\frac{c + \sqrt{c^2 - 4km}}{2m}, \quad r_2 = -\frac{c - \sqrt{c^2 - 4km}}{2m}.
\]

The general solution is

\[
    y(t) = y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad (5)
\]

This is called the overdamped case. The motion is nonoscillatory. Since

\[
    \sqrt{c^2 - 4km} < \sqrt{c^2} = c,
\]

\( r_1 \) and \( r_2 \) are both negative and \( y(t) \to 0 \) as \( t \to \infty \).
Case 3: $c^2 - 4km = 0$. In this case the characteristic equation has only one real root:

$$r_1 = -\frac{c}{2m}$$

and the general solution is

$$y(t) = y = C_1 e^{-(c/2m)t} + C_2 t e^{-(c/2m)t}.$$  \hspace{1cm} (6)

This is called the critically damped case. Once again, the motion is nonoscillatory and $y(t) \to 0$ as $t \to \infty$.  

In both the overdamped and critically damped cases, the object moves back to the equilibrium position ($y(t) \to 0$ as $t \to \infty$). The object may move through the equilibrium position once, but only once. Two typical examples of the motion are shown below.

Remark: It is important to understand the effect of damping on the motion of the spring-mass system. If $y = y(t)$ denotes the position of the mass at time $t$, then $\lim_{t \to \infty} y(t) = 0$, independent of whether the system is underdamped, overdamped, or critically damped.

III. Undamped Forced Vibrations. The vibrations that we have considered thus far result from the interplay of three forces: gravity, the restoring force of the spring, and the retarding force of friction or the surrounding medium. Such vibrations are called free vibrations.

The application of an external force to a freely vibrating system modifies the vibrations and produces what are called forced vibrations. As an example we’ll investigate the effect of a periodic external force $F_0 \cos \gamma t$ where $F_0$ and $\gamma$ are positive constants.

In an undamped system the force equation is

$$F = -kx + F_0 \cos \gamma t$$

and the equation of motion takes the form

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \cos \gamma t.$$
We set \( \omega = \sqrt{k/m} \) and write the equation of motion as
\[
y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t. \tag{7}
\]
As we’ll see, the nature of the motion depends on the relation between the applied frequency, \( \gamma/2\pi \), and the natural frequency of the system, \( \omega/2\pi \).

**Case 1:** \( \gamma \neq \omega \). In this case the method of undetermined coefficients gives the particular solution
\[
z(t) = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t
\]
and the general equation of motion is
\[
y = A \sin (\omega t + \phi_0) + \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t. \tag{8}
\]
If \( \omega/\gamma \) is rational, the vibrations are periodic. If \( \omega/\gamma \) is not rational, then the vibrations are not periodic and can be highly irregular. In either case, the vibrations are bounded by
\[
|A| + \left| \frac{F_0/m}{\omega^2 - \gamma^2} \right|. \quad \blacksquare
\]

**Case 2:** \( \gamma = \omega \). In this case the method of undetermined coefficients gives
\[
z(t) = \frac{F_0}{2\omega m} t \sin \omega t
\]
and the general solution has the form
\[
y = A \sin (\omega t + \phi_0) + \frac{F_0}{2\omega m} t \sin \omega t. \tag{9}
\]
The system is said to be in *resonance*. The motion is oscillatory but, because of the \( t \) factor in the second term, it is not periodic. As \( t \to \infty \), the amplitude of the vibrations increases without bound.
A typical illustration of the motion is given in the figure below. \( \square \)
IV. Damped Forced Vibrations. In a damped, forced system, the force equation is

\[ F = -ky - cy' + F_0 \cos \gamma t \]

and the equation that governs the motion is

\[ my'' = -ky - cy' + F_0 \cos \gamma t \]

which is written

\[ y'' + \frac{c}{m} y' + \omega^2 y = \frac{F_0}{m} \cos \gamma t \quad (10) \]

where, as before, \( \omega^2 = k/m \).

By the method of undetermined coefficients, we know that a particular solution of this equation will have the form

\[ z(t) = A \cos \gamma t + B \sin \gamma t. \]

which can also be written as

\[ z(t) = C \cos(\gamma t + \psi_0) \]

(see Problem 5 in the Exercises).

Applying the method, we find that

\[ z(t) = \frac{F_0 m (\omega^2 - \gamma^2)}{m^2 (\omega^2 - \gamma^2)^2 + c^2 \gamma^2} \cos \gamma t + \frac{F_0 c \gamma}{m^2 (\omega^2 - \gamma^2)^2 + c^2 \gamma^2} \sin \gamma t \]

\[ = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2 \gamma^2}} \cos(\gamma t + \psi_0). \]

It now follows that the general solution of (10) is

\[ y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2 \gamma^2}} \cos(\gamma t + \psi_0) \quad (11) \]

where \( y_c = C_1 e^{r_1 t} + C_2 e^{r_2 t} \) is the general solution of the reduced equation of (10) which we found in the undamped, free vibration case.

Recall that, regardless of whether the roots \( r_1, r_2 \) of the characteristic equation are real or complex numbers,

\[ \lim_{t \to \infty} y_c(t) = 0. \]

Therefore

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ y_c(t) + \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2 \gamma^2}} \cos(\gamma t + \psi_0) \right] = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2 \gamma^2}} \cos(\gamma t + \psi_0). \]

In this context, \( y_c(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \) is often called the transient solution and

\[ z(t) = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2 \gamma^2}} \cos(\gamma t + \psi_0) \]

is called the steady state solution.
Exercises 3.6

1. An object is in simple harmonic motion. Find an equation for the motion given that the period is \( \frac{1}{4}\pi \) and, at time \( t = 0, \ y = 1, \ y' = 0 \). What is the amplitude? What is the natural frequency?

2. An object is in simple harmonic motion. Find an equation for the motion given that the period is \( \pi \) and, at time \( t = 0, \ y = 0, \ y' = -2 \). What is the amplitude? What is the natural frequency?

3. An object is in simple harmonic motion with period \( T \) and amplitude \( A \). What is the velocity at the equilibrium point \( y = 0 \)?

4. An object in simple harmonic motion passes through the equilibrium point \( y = 0 \) at time \( t = 0 \) and every three seconds thereafter. Find the equation of motion given that \( y'(0) = 5 \).

5. Show that simple harmonic motion \( y(t) = C_1 \cos \omega t + C_2 \sin \omega t \) can be written as:

   (a) \( y(t) = A \sin(\omega t + \phi_0) \);  
   (b) \( y(t) = A \cos(\omega t + \psi_0) \).

6. What is the effect of an increase in the resistance constant \( c \) on the amplitude and frequency of the vibrations given by (4)?

7. Show that the motion given by (5) can pass through the equilibrium point at most once. How many times can the motion change directions?

8. Show that the motion given by (6) can pass through the equilibrium point at most once. How many times can the motion change directions?

9. Show that if \( \gamma \neq \omega \), then the method of undetermined coefficients applied to (7) gives

\[
z = \frac{F_0}{m} \frac{1}{\omega^2 - \gamma^2} \cos \gamma t.
\]

10. Show that if \( \omega/\gamma \) is rational, then the vibrations given by (8) are periodic.

11. Show that if \( \gamma = \omega \), then the method of undetermined coefficients applied to (7) gives

\[
z = \frac{F_0}{2\omega m} t \sin \omega t.
\]

12. Show that the method of undetermined coefficients applied to (10) gives

\[
z = \frac{F_0 m (\omega^2 - \gamma^2)}{m^2 (\omega^2 - \gamma^2)^2 + c^2 \gamma^2} \cos \gamma t + \frac{F_0 c \gamma}{m^2 (\omega^2 - \gamma^2)^2 + c^2 \gamma^2} \sin \gamma t.
\]

Steady-state and transient problems.
3.7 Higher-Order Linear Differential Equations

This section is a continuation of Sections 3.1 - 3.5. As you will see, all of the “theory” that we developed for second-order linear differential equations carries over, essentially verbatim, to linear differential equations of order greater than two.

Recall that a first order, linear differential equation is an equation which can be written in the form

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on some interval \( I \). A second order, linear differential equation has an analogous form.

\[ y'' + p(x)y' + q(x)y = f(x) \]

where \( p, q \), and \( f \) are continuous functions on some interval \( I \).

In general, an \( n^{th} \)-order linear differential equation is an equation that can be written in the form

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (L) \]

where \( p_0, p_1, \ldots, p_{n-1}, \) and \( f \) are continuous functions on some interval \( I \). As before, the functions \( p_0, p_1, \ldots, p^{n-1} \) are called the coefficients, and \( f \) is called the forcing function or the nonhomogeneous term.

Equation (L) is homogeneous if the function \( f \) on the right side is 0 for all \( x \in I \). In this case, equation (L) becomes

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad (H) \]

Equation (L) is nonhomogeneous if \( f \) is not the zero function on \( I \), i.e., (L) is nonhomogeneous if \( f(x) \neq 0 \) for some \( x \in I \). As in the case of second order linear equations, almost all of our attention will be focused on homogeneous equations.

Remarks on “Linear.” Intuitively, an \( n^{th} \)-order differential equation is linear if \( y \) and its derivatives appear in the equation with exponent 1 only, and there are no so-called “cross-product” terms, \( yy', yy'', yy''' \), etc.

If we set \( L[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y \), then we can view \( L \) as an “operator” that transforms an \( n \)-times continuously differentiable function \( y = y(x) \) into the continuous function

\[ L[y(x)] = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x). \]

That is, \( L : C^n(I) \rightarrow C(I) \) where \( C^n(I) \) is the vector space of \( n \)-times continuously differentiable functions on the interval \( I \).
It is easy to check that, for any two \( n \)-times differentiable functions \( y_1(x) \) and \( y_2(x) \),

\[
L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]
\]

and, for any \( n \)-times differentiable function \( y \) and any constant \( c \),

\[
L[cy(x)] = cL[y(x)].
\]

Therefore, as introduced in Section 2.1, \( L \) is a linear differential operator. This is the real reason that equation (L) is said to be a linear differential equation. ■

THEOREM 1. (Existence and Uniqueness Theorem) Given the \( n \)th-order linear equation (L). Let \( a \) be any point on the interval \( I \), and let \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) be any \( n \) real numbers. Then the initial-value problem

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x);
y(a) = \alpha_0, \ y'(a) = \alpha_1, \ldots, \ y^{(n-1)}(a) = \alpha_{n-1}
\]

has a unique solution.

Remark: We can solve any first order linear differential equation, see Section 2.1. In contrast, there is no general method for solving second or higher order linear differential equations. However, as we saw in our study of second order equations, there are methods for solving certain special types of higher order linear equations and we shall look at these later in this section. ■

Homogeneous Equations

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{H}
\]

Note first that the zero function, \( y(x) = 0 \) for all \( x \in I \), (also denoted by \( y \equiv 0 \)) is a solution of (H). As before, this solution is called the trivial solution. Obviously, our main interest is in finding nontrivial solutions.

We now establish some essential facts about homogeneous equations. The proofs are identical to those given in Section 3.2

THEOREM 2. If \( y = y(x) \) is a solution of (H) and if \( c \) is any real number, then \( u(x) = cy(x) \) is also a solution of (H).

Any constant multiple of a solution of (H) is also a solution of (H).

THEOREM 3. If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then \( u(x) = y_1(x) + y_2(x) \) is also a solution of (H).
The sum of any two solutions of (H) is also a solution of (H).

The general theorem, which combines and extends Theorems 1 and 2, is:

**THEOREM 4.** If \( y = y_1(x) \), \( y = y_2(x) \), ..., \( y = y_k(x) \) are solutions of (H), and if \( c_1, c_2, ..., c_k \) are any \( k \) real numbers, then

\[
y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)
\]

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).

Note that if \( k = n \) in the linear combination above, then the equation

\[
y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) \quad (1)
\]

has the form of a general solution of equation (H). So the question is: If \( y_1, y_2, ..., y_n \) are solutions of (H), is the expression (1) the general solution of (H)? That is, can every solution of (H) be written as a linear combination of \( y_1, y_2, ..., y_n \)? It turns out that (1) may or not be the general solution; it depends on the relation between the solutions \( y_1, y_2, ..., y_n \).

Suppose that \( y = y_1(x), y = y_2(x), ..., y = y_n(x) \) are solutions of (H). Under what conditions is (1) the general solution of (H)?

Let \( u = u(x) \) be any solution of (H) and choose any point \( a \in I \). Suppose that

\[
\alpha_0 = u(a), \; \alpha_1 = u'(a), \; \ldots, \; \alpha_{n-1} = u^{(n-1)}(a).
\]

Then \( u \) is a member of the \( n \)-parameter family (1) if and only if there are values for \( c_1, c_2, ..., c_n \) such that

\[
\begin{align*}
c_1y_1(a) + c_2y_2(a) + \cdots + c_ny_n(a) & = \alpha_0 \\
c_1y_1'(a) + c_2y_2'(a) + \cdots + c_ny_n'(a) & = \alpha_1 \\
c_1y_1''(a) + c_2y_2''(a) + \cdots + c_ny_n''(a) & = \alpha_1 \\
& \vdots \\
c_1y_1^{(n-1)}(a) + c_2y_2^{(n-1)}(a) + \cdots + c_ny_n^{(n-1)}(a) & = \alpha_{n-1}
\end{align*}
\]

According to Cramer’s rule, we are guaranteed that this pair of equations has a solution \( c_1, c_2, ..., c_n \) if

\[
\begin{vmatrix}
y_1(a) & y_2(a) & \cdots & y_n(a) \\
y_1'(a) & y_2'(a) & \cdots & y_n'(a) \\
y_1''(a) & y_2''(a) & \cdots & y_n''(a) \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)}(a) & y_2^{(n-1)}(a) & \cdots & y_n^{(n-1)}(a)
\end{vmatrix} \neq 0.
\]
Since $a$ was chosen to be any point on $I$, we conclude that (1) is the general solution of (H) if and only if

$$
\begin{vmatrix}
  y_1(x) & y_2(x) & \ldots & y_n(x) \\
  y'_1(x) & y'_2(x) & \ldots & y'_n(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  y^{(n-1)}_1(x) & y^{(n-1)}_2(x) & \ldots & y^{(n-1)}_n(x)
\end{vmatrix} \neq 0 \quad \text{for all } x \in I. \quad (2)
$$

As you know, this determinant is called the Wronskian of the solutions $y_1, y_2, \ldots, y_n$.

**THEOREM 5.** Let $y = y_1(x), y = y_2(x), \ldots, y = y_n(x)$ be solutions of equation (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ and $y_1, y_2, \ldots, y_n$ are linearly dependent.

(ii) $W(x) \neq 0$ for all $x \in I$ which implies that $y_1, y_2, \ldots, y_n$ are linearly independent and

$$
y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)
$$

is the general solution of (H).

**Example 1.** (a) The functions $y_1(x) = x, y_2(x) = x^2$ and $y_3(x) = x^3$ are each solutions of

$$
y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3}y = 0, \quad x \in I = (0, \infty). \quad \text{(verify)}
$$

Their Wronskian is:

$$
W(x) = \begin{vmatrix}
  x & x^2 & x^3 \\
  1 & 2x & 3x^2 \\
  0 & 2 & 6x
\end{vmatrix} = 2x^3 \neq 0 \quad \text{on } I.
$$

The general solution of the differential equation is $y = c_1x + c_2x^2 + c_3x^3$.

(b) The functions $y_1(x) = e^x, y_2(x) = e^{2x}$ and $y_3(x) = e^{3x}$ are each solutions of

$$
y''' - 6y'' + 11y' - 6y = 0, \quad x \in I = (-\infty, \infty). \quad \text{(verify)}
$$

Their Wronskian is:

$$
W(x) = \begin{vmatrix}
  e^x & e^{2x} & e^{3x} \\
  e^x & 2e^{2x} & 3e^{3x} \\
  e^x & 4e^{2x} & 9e^{3x}
\end{vmatrix} = 2e^{6x} \neq 0 \quad \text{on } I.
$$

The general solution of the differential equation is $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$. ■

**DEFINITION 1.** (Fundamental Set) A set of $n$ linearly independent solutions $y = y_1(x), y = y_2(x), \ldots, y = y_n(x)$ of (H) is called a fundamental set of solutions.
A set of solutions \( y_1, y_2, \ldots, y_n \) of (H) is a fundamental set if and only if
\[
W[y_1, y_2, \ldots, y_n](x) \neq 0 \quad \text{for all } x \in I.
\]

**Homogeneous Equations with Constant Coefficients**

We have emphasized that there are no general methods for solving second or higher order linear differential equations. However, there are some special cases for which solution methods do exist. Here we consider such a case, linear equations with constant coefficients. We’ll look first at homogeneous equations.

An \( n^{th} \)-order linear homogeneous differential equation with constant coefficients is an equation which can be written in the form
\[
y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0y = 0 \tag{3}
\]
where \( a_0, a_1, \ldots, a_{n-1} \) are real numbers.

We have seen that first- and second-order equations with constant coefficients have solutions of the form \( y = e^{rx} \). Thus, we’ll look for solutions of (3) of this form

If \( y = e^{rx} \), then
\[
y' = re^{rx}, \quad y'' = r^2e^{rx}, \quad \ldots, \quad y^{(n-1)} = r^{n-1}re^{rx}, \quad y^{(n)} = r^ne^{rx}.
\]
Substituting \( y \) and its derivatives into (3) gives
\[
r^n e^{rx} + a_{n-1}r^{n-1}e^{rx} + \cdots + a_1re^{rx} + a_0e^{rx} = 0
\]
or
\[
e^{rx} (r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0) = 0.
\]
Since \( e^{rx} \neq 0 \) for all \( x \), we conclude that \( y = e^{rx} \) is a solution of (3) if and only if
\[
r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0. \tag{4}
\]

**DEFINITION 2.** Given the differential equation (3). The corresponding polynomial equation
\[
p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0.
\]
is called the characteristic equation of (3); the \( n^{th} \)-degree polynomial \( p(r) \) is called the characteristic polynomial. The roots of the characteristic equation are called the characteristic roots.

Thus, we can find solutions of the equation if we can find the roots of the corresponding characteristic polynomial. Appendix 1 gives the basic facts about polynomials with real coefficients.
In Chapter 3 we proved that if \( r_1 \neq r_2 \), then \( y_1 = e^{r_1 x} \) and \( y_2 = e^{r_2 x} \) are linearly independent. We also showed that \( y_3(x) = e^{r_1 x} \) and \( y_4(x) = xe^{r_1 x} \) are linearly independent. Here is the general result.

**THEOREM 6**

1. If \( r_1, r_2, \ldots, r_k \) are distinct numbers (real or complex), then the distinct exponential functions \( y_1 = e^{r_1 x} \), \( y_2 = e^{r_2 x} \), \ldots, \( y_k = e^{r_k x} \) are linearly independent.

2. For any real number \( \alpha \) the functions \( y_1(x) = e^{\alpha x} \), \( y_2(x) = xe^{\alpha x} \), \ldots, \( y_k(x) = x^{k-1}e^{\alpha x} \) are linearly independent.

**Proof:** In each case, the Wronskian \( W[y_1, y_2, \ldots, y_k](x) \neq 0 \).

Since all of the ground work for solving linear equations with constant coefficients was established in Chapter 3, we’ll simply give some examples here. Theorem 6 will be useful in showing that our sets of solutions are linearly independent.

**Example 2.** Find the general solution of

\[
y''' + 3y'' - y' - 3y = 0
\]

given that \( r = 1 \) is a root of the characteristic polynomial.

**SOLUTION** The characteristic equation is

\[
r^3 + 3r^2 - r - 3 = 0
\]

\[
(r - 1)(r^2 + 4r + 3) = 0
\]

\[
(r - 1)(r + 1)(r + 3) = 0
\]

The characteristic roots are: \( r_1 = 1, r_2 = -1, r_3 = -3 \). The functions \( y_1(x) = e^x, y_2(x) = e^{-x}, y_3(x) = e^{-3x} \) are solutions. Since these are distinct exponential functions, the solutions form a fundamental set and

\[
y = C_1 e^{4x} + C_2 e^{-x} + C_3 e^{-3x}
\]

is the general solution of the equation. ■

**Example 3.** Find the general solution of

\[
y^{(4)} - 4y''' + 3y'' + 4y' - 4y = 0
\]

given that \( r = 2 \) is a root of multiplicity 2 of the characteristic polynomial.

**SOLUTION** The characteristic equation is

\[
r^4 - 4r^3 + 3r^2 + 4r - 4 = 0
\]

\[
(r - 2)^2(r^2 - 1) = 0
\]

\[
(r - 2)^2(r - 1)(r + 1) = 0
\]
The characteristic roots are: \( r_1 = 1, \ r_2 = -1, \ r_3 = r_4 = 2. \) The functions \( y_1(x) = e^x, \ y_2(x) = e^{-x}, \ y_3(x) = e^{2x} \) are solutions. Based on our work in Chapter 3, we conjecture that \( y_4 = xe^{2x} \) is also a solution since \( r = 2 \) is a “double” root. You can verify that this is the case. Since \( y_4 \) is distinct from \( y_1, \ y_2, \) and is independent of \( y_3, \) these solutions form a fundamental set and
\[
y = C_1 e^x + C_2 e^{-x} + C_3 e^{2x} + C_4 xe^{2x}
\]
is the general solution of the equation.

**Example 4.** Find the general solution of
\[
y^{(4)} - 2y''' + y'' + 8y' - 20y = 0
\]
given that \( r = 1 + 2i \) is a root of the characteristic polynomial.

**SOLUTION** The characteristic equation is
\[
p(r) = r^4 - 2r^3 + r^2 + 8r - 20 = 0.
\]
Since \( 1 + 2i \) is a root of \( p(r), \ 1 - 2i \) is also a root, and \( r^2 - 2r + 5 \) is a factor of \( p(r). \)

Therefore
\[
\begin{align*}
r^4 - 2r^3 + r^2 + 8r - 20 &= 0 \\
(r^2 - 2r + 5)(r^2 - 4) &= 0 \\
(r^2 - 2r + 5)(r - 2)(r + 2) &= 0
\end{align*}
\]

The characteristic roots are: \( r_1 = 1 + 2i, \ r_2 = 1 - 2i, \ r_3 = 2, \ r_4 = -2. \) Since these roots are distinct, the corresponding exponential functions are linearly independent. Again based on our work in Chapter 3, we convert the complex exponentials
\[
\begin{align*}
u_1 &= e^{(1+2i)x} \quad \text{and} \quad u_2(x) = e^{(1-2i)x} \quad \text{into} \quad y_1 = e^x \cos 2x \quad \text{and} \quad y_2 = e^x \sin 2x.
\end{align*}
\]
Then, \( y_1, \ y_2, \ y_3 = e^{2x}, \ y_4 = e^{-2x} \) form a fundamental set and
\[
y = C_1 e^x \cos 2x + C_2 e^x \sin 2x + C_3 e^{2x} + C_4 e^{-2x}
\]
is the general solution of the equation.

**Recovering a Homogeneous Differential Equation from Its Solutions**

Once you understand the relationship between the homogeneous equation, the characteristic equation, the roots of the characteristic equation and the solutions of the differential equation, it is easy to go from the differential equation to the solutions and from the solutions to the differential equation. Here are some examples.
Example 5. Find a fourth order, linear, homogeneous differential equation with constant coefficients that has the functions $y_1(x) = e^{2x}$, $y_2(x) = e^{-3x}$ and $y_3(x) = e^{2x} \cos x$ as solutions.

**SOLUTION** Since $e^{2x}$ is a solution, 2 must be a root of the characteristic equation and $r - 2$ must be a factor of the characteristic polynomial; similarly, $e^{-3x}$ a solution means that $-3$ is a root and $r - (-3) = r + 3$ is a factor of the characteristic polynomial. The solution $e^{2x} \cos x$ indicates that $2 + i$ is a root of the characteristic equation. So $2 - i$ must also be a root (and $y_4(x) = e^{2x} \sin x$ must also be a solution). Thus the characteristic equation must be

$$(r - 2)(r + 3)(r - (2 + i))(r - (2 - i)) = (r^2 + r - 6)(r^2 - 4r + 5) = r^4 - 3r^3 - 5r^2 + 29r - 30 = 0.$$ 

Therefore, the differential equation is

$$y^{(4)} - 3y''' - 5y'' + 29y' - 30y = 0.$$

Example 6. Find a third order, linear, homogeneous differential equation with constant coefficients that has

$$y = C_1 e^{-4x} + C_2 x e^{-4x} + C_3 e^{2x}$$

as its general solution.

**SOLUTION** Since $e^{-4x}$ and $xe^{-4x}$ are solutions, $-4$ must be a double root of the characteristic equation; since $e^{2x}$ is a solution, 2 is a root of the characteristic equation. Therefore, the characteristic equation is

$$(r + 4)^2(r - 2) = 0$$

which expands to

$$r^3 + 6r^2 - 32 = 0$$

and the differential equation is

$$y''' + 6y'' - 32y = 0.$$ 

**Nonhomogeneous Equations**

Now we'll consider linear nonhomogeneous equations:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)$$

where $p_0$, $p_1$, $\ldots$, $p_{n-1}$, $f$ are continuous functions on an interval $I$.

Continuing the analogy with second order linear equations, the corresponding homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0. \quad (H)$$

is called the reduced equation of equation (N).
The following theorems are exactly the same as Theorems 1 and 2 in Section 3.4, and exactly the same proofs can be used.

**THEOREM 7** If \( z = z_1(x) \) and \( z = z_2(x) \) are solutions of (N), then

\[
y(x) = z_1(x) - z_2(x)
\]

is a solution of equation (H).

*the difference of any two solutions of the nonhomogeneous equation (N) is a solution of its reduced equation (H)*.

The next theorem gives the “structure” of the set of solutions of (N).

**THEOREM 8** Let \( y = y_1(x), y_2(x), \ldots, y_n(x) \) be a fundamental set of solutions of the reduced equation (H) and let \( z = z(x) \) be a particular solution of (N). If \( u = u(x) \) is any solution of (N), then there exist constants \( c_1, c_2, \ldots, c_n \) such that

\[
u(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + z(x)
\]

According to Theorem 8, if \( \{y_1(x), y_2(x), \ldots, y_n(x)\} \) is a fundamental set of solutions of the reduced equation (H) and if \( z = z(x) \) is a particular solution of (N), then

\[
y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + z(x)
\]

(5)
represents the set of all solutions of (N). That is, (5) is the general solution of (N). Another way to look at (5) is: The general solution of (N) consists of the general solution of the reduced equation (H) plus a particular solution of (N):

\[
y = \underbrace{C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)}_{\text{general solution of (H)}} + \underbrace{z(x)}_{\text{particular solution of (N)}}.
\]

The superposition principle also holds:

**THEOREM 9** If \( z = z_f(x) \) and \( z = z_g(x) \) are particular solutions of

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x),
\]

and

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = g(x)
\]

respectively, then \( z(x) = z_f(x) + z_g(x) \) is a particular solution of

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) + g(x).
\]
Finding a Particular Solution

The method of variation of parameters can be extended to higher-order linear nonhomogeneous equations but the calculations become quite involved. Instead we’ll look at the special equations for which the method of undetermined coefficients can be used.

As we saw in Chapter 3, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form

\[ y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0y = f(x), \]

where \( a_0, a_1, \ldots, a_{n-1} \) are constants and the nonhomogeneous term \( f \) is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.

Here is the basic table from Section 3.5, slightly modified to apply to equations of order greater than 2:

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A particular solution of ( y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x) )</td>
</tr>
<tr>
<td>If ( f(x) = ce^{rx} ) then try ( z(x) = Ae^{rx} )</td>
</tr>
<tr>
<td>If ( f(x) = c \cos \beta x + d \sin \beta x ) then try ( z(x) = A \cos \beta x + B \sin \beta x )</td>
</tr>
<tr>
<td>If ( f(x) = ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x ) then try ( z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x )</td>
</tr>
</tbody>
</table>

*Note: If \( z \) satisfies the reduced equation, then \( x^k z \), where \( k \) is the least integer such that \( x^k z \) does not satisfy the reduced equation, will give a particular solution

The method of undetermined coefficients is applied in exactly the same manner as in Section 3.5.

**Example 7.** Find the general solution of

\[ y''' - 2y'' - 5y' + 6y = 4 - 2e^{2x}. \]  

\(*\)

**SOLUTION** First we solve the reduced equation

\[ y''' - 2y'' - 5y' + 6y = 0. \]

The characteristic equation is

\[ r^3 - 2r^2 - 5r + 6 = (r - 1)(r + 2)(r - 3) = 0. \]
The roots are \( r_1 = 1, \ r_2 = -2, \ r_3 = 3 \) and the corresponding solutions of the reduced equation are \( y_1 = e^x, \ y_2 = e^{-2x}, \ y_3 = e^{3x} \). Since these are distinct exponential functions, they are linearly independent and

\[ y = C_1 e^x + C_2 e^{-2x} + C_3 e^{3x} \]

is the general solution of the reduced equation.

Next we find a particular solution of the nonhomogeneous equation. The table indicates that we should look for a solution of the form

\[ z = A + Be^{2x} \]

The derivatives of \( z \) are:

\[ z = A + Be^{2x}, \quad z' = 2Be^{2x}, \quad z'' = 4Be^{2x}, \quad z''' = 8Be^{2x}. \]

Substituting into the left side of \((*)\), we get

\[
\begin{align*}
z''' - 2z'' - 5z' + 6z &= 8Be^{2x} - 2(4Be^{2x}) - 5(2Be^{2x}) + 6(A + Be^{2x}) \\ &= 6A - 4Be^{2x}.
\end{align*}
\]

Setting \( z'' + 6z' + 9z = 4 - 2e^{2x} \) gives

\[
6A = 4 \quad \text{and} \quad -4B = -2 \quad \text{which implies} \quad A = \frac{2}{3} \quad \text{and} \quad B = \frac{1}{2}.
\]

Thus, \( z(x) = \frac{2}{3} + \frac{1}{2} e^{2x} \) is a particular solution of \((*)\).

The general solution of \((*)\) is

\[
y = C_1 e^x + C_2 e^{-2x} + C_3 e^{3x} + \frac{2}{3} + \frac{1}{2} e^{2x}. \quad \blacksquare
\]

**Example 8.** Find the general solution of

\[
y^{(4)} + y''' - 3y'' - 5y' - 2y = 6e^{-x} \quad (**)\]

**SOLUTION** First we solve the reduced equation

\[
y^{(4)} + y''' - 3y'' - 5y' - 2y = 0.
\]

The characteristic equation is

\[
r^4 + r^3 - 3r^2 - 5r - 2 = (r + 1)^3(r - 2) = 0.
\]

The roots are \( r_1 = r_2 = r_3 = -1, \ r_4 = 2 \) and the corresponding solutions of the reduced equation are \( y_1 = e^{-x}, \ y_2 = xe^{-x}, \ y_3 = x^2e^{-x}, \ y_4 = e^{2x} \). Since distinct powers of \( x \) are linearly independent, it follows that \( y_1, \ y_2, \ y_3 \) are linearly independent; and since \( e^{2x} \)
and \( e^{-x} \) are independent, we can conclude that \( y_1, y_2, y_3, y_4 \) are linearly independent. Thus, the general solution of the reduced equation is

\[
y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} + C_4 e^{2x}.
\]

Next we find a particular solution of the nonhomogeneous equation. The table indicates that we should look for a solution of the form

\[
z = Ax^3 e^{-x}.
\]

The derivatives of \( z \) are:

\[
\begin{align*}
z &= Ax^3 e^{-x} \\
z' &= 3Ax^2 e^{-x} - Ax^3 e^{-x} \\
z'' &= 6Axe^{-x} - 6Ax^2 e^{-x} + Ax^3 e^{-x} \\
z''' &= 6Ae^{-x} - 18Axe^{-x} + 9Ax^2 e^{-x} - Ax^3 e^{-x} \\
z^{(4)} &= -24e^{-x} + 36Axe^{-x} - 12Ax^2 e^{-x} + Ax^3 e^{-x}
\end{align*}
\]

Substituting \( z \) and its derivatives into the left side of (**), we get

\[
z^{(4)} + 3z''' - 5z'' - 2z = -18Ae^{-x}.
\]

Thus, we have \(-18Ae^{-x} = 6e^{-x}\) which implies \( A = -\frac{1}{3}\) and \( z = -\frac{1}{3} x^2 e^{-x}\) is a particular solution of (**).

The general solution of (**) is

\[
y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} + C_4 e^{2x} - \frac{1}{3} x^3 e^{-x}.
\]

Example 9. Give the form of a particular solution of

\[
y'' - 3y'' + 3y' - y = 4e^x - 3 \cos 2x.
\]

SOLUTION To get the proper form for a particular solution of the equation we need to find the solutions of the reduced equation:

\[
y'' - 3y'' + 3y' - y = 0.
\]

The characteristic equation is

\[
r^3 - 3r^3 + 3r - 1 = (r - 1)^3 = 0.
\]

Thus, the roots are \( r_1 = r_2 = r_3 = 1 \), and the corresponding solutions are \( y_1 = e^x, y_2 = xe^x, y_3 = x^2 e^x \). The table indicates that the form of a particular solution \( z \) of the nonhomogeneous equation is

\[
z = Ax^3 e^x + B \cos 2x + C \sin 2x.
\]
Example 10. Give the form of a particular solution of

\[ y^{(4)} - 16y = 4e^{2x} - 2e^{3x} + 5 \sin 2x + 2 \cos 2x. \]

SOLUTION To get the proper form for a particular solution of the equation we need to find the solutions of the reduced equation:

\[ y^{(4)} - 16y = 0. \]

The characteristic equation is

\[ r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r^2 + 4) = 0. \]

Thus, the roots are \( r_1 = 2, r_2 = -2, r_3 = 2i, r_4 = -2i, \) and the corresponding solutions are \( y_1 = e^{2x}, y_2 = e^{-2x}, y_3 = \cos 2x, y_4 = \sin 2x. \) The table indicates that the form of a particular solution \( z \) of the nonhomogeneous equation is

\[ z = Axe^{2x} + Be^{3x} + Cx \cos 2x + Dx \sin 2x. \]

Exercises 3.7

Find the general solution of the homogeneous equation

1. \( y''' - 6y'' + 11y' - 6y = 0, \) \( r_1 = 1 \) is a root of the characteristic equation.
2. \( y''' + y' + 10y = 0, \) \( r_1 = -2 \) is a root of the characteristic equation.
3. \( y^{(4)} - 2y''' + y'' + 8y' - 20y = 0, \) \( r_1 = 1 + 2i \) is a root of the characteristic equation.
4. \( y^{(4)} - 3y''' - 4y = 0, \) \( r_1 = i \) is a root of the characteristic equation.
5. \( y^{(4)} - 4y''' + 14y'' - 4y' + 13y = 0, \) \( r_1 = i \) is a root of the characteristic equation.
6. \( y''' + y' - 4y' - 4y = 0, \) \( r_1 = -1 \) is a root of the characteristic equation.
7. \( y^{(6)} - y'' = 0. \)
8. \( y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0. \)

Find the solution of the initial-value problem.

9. \( y^{(4)} - 4y''' + 4y'' = 0; \) \( y(0) = -1, y'(0) = 2, y''(0) = 0, y'''(0) = 0. \)
10. \( y''' + y' = 0; \) \( y(0) = 0, y'(0) = 1, y''(0) = 2. \)
11. \( y''' - y' + 9y' - 9y = 0; \) \( y(0) = y'(0) = 0, y''(0) = 2. \)
12. \(2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0;\) \(y(0) = 0,\) \(y'(1) = 2,\) \(y''(0) = 2,\) \(y'''(0) = 0.\)

Find the homogeneous equation with constant coefficients that has the given general solution.

13. \(y = C_1 e^{-3x} + C_2 e^{-3x} + C_3 e^x \cos 3x + C_4 e^x \sin 3x.\)

14. \(y = C_1 e^{4x} + C_2 x + C_3 + C_4 e^x \cos 2x + C_5 e^x \sin 2x.\)

15. \(y = C_1 e^{3x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + C_5.\)

16. \(y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4.\)

Find the homogeneous equation with constant coefficients of least order that has the given function as a solution.

17. \(y = 2e^{2x} + 3 \sin x - x.\)

18. \(y = 3xe^{-x} + e^{-x} \cos 2x + 1.\)

19. \(y = 2e^x - 3e^{-x} + 2x.\)

20. \(y = 3e^{3x} - 2 \cos 2x + 4 \sin x - 3.\)

Find the general solution of the nonhomogeneous equation.

21. \(y''' + y'' + y' + y = e^x + 4.\)

22. \(y^{(4)} - y = 2e^x + \cos x.\)

23. \(y^{(4)} + 2y'' + y = 6 + \cos 2x.\)

24. \(y''' - y'' - y' + y = 2e^{-x} + 4e^{2x}.\)

Find the solution of the initial-value problem.

25. \(y''' - 2y'' - 5y' + 6y = 2e^x(1 - 6x);\) \(y(0) = 2, y'(0) = 7, y''(0) = 9.\)

26. \(y''' - 2y'' - 5y' + 6y = 2e^x;\) \(y(0) = 2, y'(0) = 0, y''(0) = -1.\)

Give the form of a particular solution of the nonhomogeneous equation.

27. \(y''' - y'' - y' + y = 2e^{-x} + 3e^x + 2,\) \(r = -1\) is a root of the characteristic equation.

28. \(y''' + y'' - 2y = -2e^x + 3 \cos 2x,\) \(r = 1\) is a root of the characteristic equation.

29. \(y''' - 6y'' + 11y' - 6y = 7e^{2x} - 4e^{-x} - 2 \sin 5x,\) \(r = 3\) is a root of the characteristic equation.

30. \(y''' - 2y'' + y' = 2e^{2x} - e^{-2x} + 4.\)
31. $y''' - 4y'' + 5y' - 2y = e^x + 3e^{2x} + e^{2x} \sin 3x$, $r = 1$ is a root of the characteristic equation.

32. $y^{(4)} + 5y'' - 36y = 7e^{-2x} + e^{2x} + 2 \cos 2x$.

33. $y^{(4)} - 2y'' + y = (3x + 1)e^{-x} - 5e^{2x} + 2x$

34. $y^{(4)} - 3y''' + 4y'' - 2y' = 5x + 3e^{-x} + 2e^{x} \cos 2x$, $r = 1$ is a root of the characteristic equation.

35. $y^{(4)} - 3y''' + 2y'' + 2y' - 4y = -3e^{2x} + 2xe^{-x} + e^{x} \cos x$, $r = 2$ is a root of the characteristic equation.

36. $y^{(4)} + y''' + 2y'' + 2y' - 8y = 2xe^x + 3e^{2x} + 4x$, $r = 1$ is a root of the characteristic equation.
CHAPTER 3. Answers to Odd-Numbered Problems

Exercises 3.2

1. Yes
2. Yes
3. Yes
4. Yes
5. (a) $r = -1, r = 4$.
6. Fundamental set: $y_1(x) = x^{-1}, y_2(x) = x^4$; general solution: $y = C_1 x^{-1} + C_2 x^4$.
7. (b) $y = \frac{2}{3} x^{-1} + \frac{1}{3} x^4$.
8. The trivial solution: $y \equiv 0$.
9. $y'' - 2y' - 3y = 0$.
10. $y'' = 0$.
11. $x^2 y'' - 2x y' + 2y = 0$.
12. $W[y_1, y_2](x) = e^{-\int p(t) dt} \neq 0$ for all $x$.
13. $\{y_1(x) = e^{3x}, y_2(x) = xe^{3x}\}$.
14. $\{y_1(x) = x, y_2(x) = x^2\}$.
15. $\{y_1(x) = x^2, y_2(x) = x^2 \ln x\}$.
16. $\{y_1(x) = e^{x^2}, y_2(x) = e^{-x^2}\}$.
17. Set $u(x) = \frac{y_2(x)}{y_1(x)}$. Then

$$u'(x) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2} \equiv 0.$$ 

Therefore, $u \equiv \lambda$ constant, which implies that $y_2 = \lambda y_1$.
18. $\alpha \delta - \beta \gamma \neq 0$.
19. $W[y_1 + y_2, y_1 - y_2] = -2W[y_1, y_2]$.

Exercises 3.3

1. $y = C_1 e^{2x} + C_2 e^{-4x}$.
2. $y = C_1 e^{5x} + C_2 x e^{5x}$.
3. $y = e^{-2x} [C_1 \cos 3x + C_2 \sin 3x]$.
4. $y = C_1 e^{x/2} + C_2 e^{-3x}$.
5. $y = C_1 \cos 3x + C_2 \sin 3x$.
6. $y = C_1 e^{3x/2} + C_2 x e^{3x/2}$.
13. \( y = C_1 e^{6x} + C_2 e^{-5x} \).
15. \( y = e^{3x} \left[ C_1 \cos x + C_2 \sin x \right] \).
17. \( y = 2e^{2x} - e^{3x} \).
19. \( y = -3e^{-x} - 2xe^{-x} \).
21. \( y = -e^x \cos x \).
23. \( y'' + 3y' - 10y = 0 \).
25. \( y'' + 4y = 0 \).
27. \( y'' + y' - 12y = 0 \).
29. \( y'' - y' + \frac{1}{4}y = 0 \).
31. \( y = \frac{\alpha + 2}{3} e^{2x} + \frac{2\alpha - 2}{3} e^{-x}; \quad \alpha = -2 \).
33. \( (a) \ a < -1/2 \quad (b) \ a > 1/2 \)
35. If the roots of \( r^2 + ar + b = 0 \) are real (real and unequal, or real and equal), then they are negative; \( r \) negative implies \( e^{rx} \to 0 \) and \( xe^{rx} \to 0 \) as \( x \to \infty \). If the roots are complex conjugates, then they have negative real part and \( \alpha \) negative implies \( e^{\alpha x} \cos \beta x \to 0 \) and \( e^{\alpha x} \sin \beta x \to 0 \) as \( x \to \infty \).
39. \( W [e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x] = \beta e^{2\alpha x} \)
41. \( y = C_1 x + C_2 x^2 \).
43. \( y = C_1 x^{-3} + C_2 x^{-3} \ln x \).
45. \( y = C_1 x^2 \cos 4 \ln x + C_2 x^2 \sin 4 \ln x \).

Exercises 3.4

1. \( z(x) = x^2 \ln x + \frac{1}{2}; \quad y = C_1 x^2 + C_2 x^{-1} + x^2 \ln x + \frac{1}{2} \).
3. \( z(x) = -2x^2; \quad y = C_1 e^{x^2} + C_2 e^{-x^2} - 2x^2 \).
5. \( z(x) = -(1 + x^2); \quad y = C_1 x + C_2 e^x - (1 + x^2) \).
7. \( z(x) = \frac{1}{2}x \sin x; \quad y = C_1 x \cos x + C_2 x \sin x + \frac{1}{2}x^2 \sin x \).
9. \( y = C_1 e^{-x} + C_2 e^{2x} - \frac{2}{3}x e^{-x} \).
11. \( y = C_1 e^{2x} + C_2 xe^{2x} + 2xe^{2x} \).
13. \( y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \cos 2x \ln (\cos 2x) + \frac{1}{2}x \sin 2x \).
15. \( y = C_1 e^x + C_2 xe^x - e^x \cos x \).
17. \( y = C_1 e^{-2x} + C_2 xe^{-2x} - e^{-2x} \ln x \).
19. \( y = C_1 \cos 3x + C_2 \sin 3x + \sin 3x \ln (\sec 3x + \tan 3x) - 1 \).
21. \( y = C_1 x + C_2 x^2 - x^2 \ln x + \frac{1}{2}x^2 (\ln x)^2 \).
23. \( y = C_1 x + C_2 (x^2 - 1) + \frac{1}{2}x^2 + \frac{1}{6}x^4 \).
25. \( y = C_1 x + C_2 x \ln x + x^2 \).

Exercises 3.5

1. \( y = C_1 e^{-x} + C_2 e^{3x} - e^{2x} \).
2. \( y = C_1 e^{-3x} + C_2 e^{-3x} + \frac{1}{4} e^{3x} \).
3. \( y = C_1 e^{-2x} + C_2 - \frac{1}{2} \cos 2x - \frac{1}{2} \sin 2x \).
4. \( y = C_1 e^{3x} + C_2 x e^{3x} + \frac{1}{6} e^{-3x} \).
5. \( y = C_1 e^{-2x} + C_2 e^{-4x} + \frac{2}{3} x e^{-2x} \).
6. \( y = e^x (C_1 \cos 2x + C_2 \sin 2x) - \frac{1}{10} e^{-x} \cos 2x - \frac{1}{20} e^{-x} \sin 2x \).
7. \( y = -\frac{19}{30} \cos 2x + \frac{7}{10} \sin 2x + \frac{1}{4} x^2 - \frac{1}{8} + \frac{3}{5} e^x \).
8. \( z = A + (Bx^2 + Cx)e^{-x} + D \cos 3x + E \sin 3x \).
9. \( z = Ax^2 + Bx + C + Dx \cos x + Ex \sin x \).
10. \( z = (Ax^3 + Bx^2)e^{2x} + Cx^2 + Dx + E + (Fx + G) \cos 2x + (Hx + I) \sin 2x \).
11. \( z = Ae^{-x} + Bxe^{-x} \cos x + Cxe^{-x} \sin x + D \).
12. \( y = C_1 e^{2x} + C_2 xe^{2x} + \frac{8}{25} \cos x + \frac{6}{25} \sin x + 3xe^{2x} \ln x \).
13. \( y = C_1 \cos 3x + C_2 \sin 3x + \frac{3}{8} \cos x - \sin 3x \ln(\sec 3x + \tan 3x) + 1 \).

Exercises 3.6

1. The equation of motion is \( y(t) = \sin(8t + \frac{1}{2}\pi) \). The amplitude is 1 and the frequency is \( 8/2\pi = 4/\pi \).
2. The velocity at the equilibrium point is: \( \pm 2\pi A/T \).
3. (a) \( A \sin(\omega t + \phi_0) = A \cos(\omega t + \phi_0 - \frac{\pi}{2}) \); take \( \phi_1 = \phi_0 - \frac{\pi}{2} \).
   (b) \( A \sin(\omega t + \phi_0) = A \cos \phi_0 \sin \omega t + A \sin \phi_0 \cos \omega t = B \sin \omega t + C \cos \omega t \).
4. Assume that \( r_1 > r_2 \). If \( C_1 = 0 \) or \( C_2 = 0 \), then \( y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \) can never be zero. If both \( C_1 \) and \( C_2 \) are nonzero, then \( C_1 e^{r_1 t} + C_2 e^{r_2 t} = 0 \) implies \( e^{(r_1-r_2)t} = -\frac{C_2}{C_1} \). Since \( e^{(r_1-r_2)t} \) is an increasing function \( (r_1 > r_2) \), it can take the value \( \frac{C_2}{C_1} \) at most once.
   By the same reasoning, \( x'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} \) can be zero at most once. Therefore the motion can change direction at most once.
9. If \( \gamma \neq \omega \), we try \( z = A \cos \gamma t + B \sin \gamma t \) as a particular solution of \( y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t \).

Substituting \( z \) into the equation, we get \( -\gamma^2 z + \omega^2 z = \frac{F_0}{m} \cos \gamma t \), giving

\[
z = \frac{F_0/m}{\omega^2 - \gamma^2 \cos \gamma t}.
\]

11. If \( \gamma = \omega \), we try \( z = At \cos \omega t + Bt \sin \omega t \) as a particular solution of \( y'' + \omega^2 y = \frac{F_0}{m} \cos \omega t \).

Substituting \( z \) into the equation, we have

\[
(2B\omega - A\omega^2 t) \cos \omega t - (2A\omega + B\omega^2 t) \sin \omega t + \omega^2 (At \cos \omega t + Bt \sin \omega t) = \frac{F_0}{m} \cos \omega t,
\]

which gives \( A = 0 \), \( B = \frac{F_0}{2\omega m} \), as required.

Exercises 3.7

1. \( y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x} \).

3. \( y = C_1 e^{2x} + C_2 e^{-2x} + e^x [C_3 \cos 2x + C_4 \sin 2x] \).

5. \( y = C_1 \cos x + C_2 \sin x + e^{2x} [C_3 \cos 3x + C_4 \sin 3x] \).

7. \( y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} + C_5 \cos x + C_6 \sin x \).

9. \( y = 2x - 1 \).

11. \( y = \frac{1}{6} e^x - \frac{1}{5} \cos 3x - \frac{1}{15} \sin 3x \).

13. \( y^{(4)} - 8y''' + 31y'' - 78y' + 90y = 0 \).

15. \( y^{(5)} - 2y^{(4)} - 2y''' - 2y'' - 3y' = 0 \).

17. \( y^{(5)} - 2y^{(4)} + y''' - 2y'' = 0 \).

19. \( y^{(4)} - y'' = 0 \).

21. \( y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x + \frac{1}{4} e^x + 4 \).

23. \( y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x + 6 + \frac{1}{9} \cos 2x \).

25. \( y = (x^2 + 2)e^x = e^{-2x} + e^{3x} \).

27. \( z = A x e^{-x} + B x^2 e^x + C \).

29. \( z = A e^{2x} + B e^{-x} + C \cos 5x + D \sin 5x \).

31. \( z = A x^2 e^x + B x e^{2x} + C e^{2x} \cos 3x + D e^{2x} \sin 3x \).

33. \( z = (A x^3 + B x^2) e^{-x} + C e^{2x} + D x + E \).

35. \( z = A e^{2x} + (B x^2 + C x) e^{-x} + D x e^x \cos x + E x e^x \sin x \).