1. Basic Terminology

A differential equation is an equation that contains an unknown function together with one or more of its derivatives.
Examples:

1. \( y' = 2x + \cos x \)

2. \( \frac{dy}{dt} = ky \) (exponential growth/decay)

3. \( x^2y'' - 2xy' + 2y = 4x^3 \)
4. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \] (Laplace’s eqn.)

5. \[ \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 0 \]
**TYPE:**

If the unknown function depends on a single independent variable, then the equation is an **ordinary differential equation** (ODE).

If the unknown function depends on more than one independent variable, then the equation is a **partial differential equation** (PDE).
ORDER:

The order of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.
Examples:

1. \[ y' = 2x + \cos x \]

2. \[ \frac{dy}{dt} = ky \text{ (exponential growth/decay)} \]

3. \[ x^2y'' - 2xy' + 2y = 4x^3 \]
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace’s eqn.)

5. $\frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} = 0$

6. $\frac{d^2 y}{dx^2} + 2x \sin \left( \frac{dy}{dx} \right) + 3e^{xy} = \frac{d^3}{dx^3}(e^{2x})$
SOLUTION:

A solution of a differential equation is a function defined on some domain $D$ such that the equation reduces to an identity when the function is substituted into the equation.
Examples:

1. \( y' = 2x + \cos x \)
2. \( y' = 3y \)
3. \[ y'' - 2y' - 8y = 4e^{2x} \]

Is \( y = 2e^{4x} - \frac{1}{2}e^{2x} \) a solution?
\[ y'' - 2y' - 8y = 4e^{2x} \]

Is \( y = e^{-2x} + 2e^{3x} \) a solution?
4. \( x^2 y'' - 4xy' + 6y = 3x^4 \)

Is \( y = \frac{3}{2} x^4 + 2x^3 \) a solution?
\[ x^2 y'' - 4xy' + 6y = 3x^4 \]

Is \( y = 2x^2 + x^3 \) a solution?
5. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u = \ln \sqrt{x^2 + y^2} \quad \text{Solution?} \]
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

\[u = \cos x \sinh y, \quad u = 3x - 4y\]

Solutions??
6. Find a value of \( r \), if possible, such that \( y = e^{rx} \) is a solution of
\[
y'' - 3y' - 10y = 0.
\]
7. Find a value of $r$, if possible, such that $y = x^r$ is a solution of

$$x^2 y'' + 2x y' - 6y = 0.$$
8. Find a value of \( r \), if possible, such that \( y = x^r \) is a solution of

\[
y'' - \frac{1}{x} y' - \frac{3}{x^2} y = 0.
\]
From now on, all differential equations are ordinary differential equations.
2. \( n \)-PARAMETER FAMILY OF SOLUTIONS / GENERAL SOLUTION

Example: Solve the differential equation:

\[
y''' - 12x + 6e^{2x} = 0
\]
NOTE: To solve a differential equation having the special form

\[ y^{(n)}(x) = f(x), \]

simply integrate \( f \) \( n \) times,

and EACH integration step produces an arbitrary constant.
Intuitively, to find a set of solutions of an $n$-th order differential equation

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0$$

we “integrate” $n$ times, with each integration step producing an arbitrary constant of integration (i.e., a parameter). Thus, ”in theory,” an $n$-th order differential equation has an $n$-parameter family of solutions.
SOLVING A DIFFERENTIAL EQUATION:

To solve an $n$-th order differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

means to find an $n$-parameter family of solutions. (Note: Same $n$.)

Note: An “$n$-parameter family of solutions” is more commonly called the GENERAL SOLUTION.
Examples: Find the general solution:

1. \( y' - 3x^2 - 2x + 4 = 0 \)
\[ y = x^3 + x^2 - 4x + C \]
2. \( y'' + 2 \sin 2x = 0 \)
\[ y = \frac{1}{2} \sin 2x + C_1 x + C_2 \]
3. \( y''' - 3y'' + 3y' - y = 0 \)

Answer: \( y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x \)

4. \( x^2 y'' - 4xy' + 6y = 3x^4 \)

Answer: \( y = C_1 x^2 + C_2 x^3 + \frac{3}{2} x^4 \)
PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a particular solution of the equation.
Examples:

1. \[ y'' = 6x + 8e^{2x} \]

General solution:

\[ y = x^3 + 2e^{2x} + C_1x + C_2 \]

Particular solutions:
2. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

General solution:

\[ y = C_1x + C_2x^2 + 2x^3 \]

Particular solutions:
3. THE DIFFERENTIAL EQUATION OF AN $n$-PARAMETER FAMILY:

Given an $n$-parameter family of curves. The differential equation of the family is an $n$-th order differential equation that has the given family as its general solution.
Examples:

1. $y^2 = Cx^3 + 4$ is the general solution of a DE.

a. What is the order of the DE?

b. Find the DE?
2. \( y = C_1 x + C_2 x^3 \) is the general solution of a DE.

a. What is the order of the DE?

b. Find the DE?
General strategy for finding the differential equation

**Step 1.** Differentiate the family $n$ times. This produces a system of $n + 1$ equations.

**Step 2.** Choose any $n$ of the equations and solve for the parameters.

**Step 3.** Substitute the “values” for the parameters in the remaining equation.
Examples:

The given family of functions is the general solution of a differential equation.

(a) What is the order of the equation?

(b) Find the equation.
1. \[ y = Cx^3 - 2x \]

(a) 

(b)
2. \[ y = C_1 e^{2x} + C_2 e^{3x} \]

(a) 

(b)
3. \[ y = C_1 \cos 3x + C_2 \sin 3x \]

(a)

(b)
4. \[ y = C_1 x^4 + C_2 x + C_3 \]

(a) 

(b)
5. \[ y = C_1 + C_2x + C_3x^2 \]

(a)

(b)
4. INITIAL-VALUE PROBLEMS:

1. Find a solution of

\[ y' = 3x^2 + 2x + 1 \]

which passes through the point \((-2, 4)\).
$y = x^3 + x^2 + x + C$
$$y = x^3 + x^2 + x + 10$$
2. \( y = C_1 \cos 3x + C_2 \sin 3x \) is the general solution of

\[ y'' + 9y = 0. \]

a. Find a solution which satisfies

\[ y(0) = 3 \]
b. Find a solution which satisfies

\[ y(0) = 3, \quad y'(0) = 4 \]
\[ y = 3 \cos 3x + \frac{4}{3} \sin 3x \] is the solution of

\[ y'' + 9y = 0, \quad y(0) = 3, \quad y'(0) = 4. \]
c. Find a solution which satisfies

\[ y(0) = 4, \ y(\pi) = 4 \]

d. Find a solution which satisfies

\[ y(0) = 4, \ y(\pi) = -4 \]
An \( n \)-th order initial-value problem consists of an \( n \)-th order differential equation

\[
F \left[ x, y, y', y'', \ldots, y^{(n)} \right] = 0
\]

together with \( n \) (initial) conditions of the form

\[
y(c) = k_0, \quad y'(c) = k_1, \quad y''(c) = k_2, \quad \ldots,
\]

\[
y^{(n-1)}(c) = k_{n-1}
\]

where \( c \) and \( k_0, k_1, \ldots, k_{n-1} \) are given numbers.
NOTES:

1. An $n$-th order differential equation can always be written in the form

$$F \left[ x, y, y', y'', \cdots, y^{(n)} \right] = 0$$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.
Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.
Examples:

1. Find a solution of the initial-value problem

\[ y' = 4x + 6e^{2x}, \quad y(0) = 5 \]
2. \( y = C_1 e^{-2x} + C_2 e^{4x} \) is the general solution of

\[
y'' - 2y' - 8y = 0
\]

Find a solution that satisfies the initial conditions

\[
y(0) = 3, \quad y'(0) = 2
\]
3. \( y = C_1 x + C_2 x^3 \) is the general solution of

\[
y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0
\]

a. Find a solution which satisfies

\[
y(1) = 2, \quad y'(1) = -4.
\]
b. Find a solution which satisfies

\[ y(0) = 0, \quad y'(0) = 2. \]


c. Find a solution which satisfies

\[ y(0) = 4, \quad y'(0) = 3. \]
EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem have a solution? That is, do solutions to the problem exist?

2. If a solution does exist, is it unique? That is, is there exactly one solution to the problem or is there more than one solution?