DIFFERENTIAL EQUATIONS

Background material: Text, Section 1.1

1. BASIC TERMINOLOGY (Text, Section 1.2)

A differential equation is an equation that contains an unknown function together with one or more of its derivatives.
Examples: In 1 – 4, find $y(x)$ such that:

1. $y' = 2x + \cos x$

2. $\frac{dy}{dx} = ky$ (exponential growth/decay)

3. $x^2y'' - 2xy' + 2y = 4x^3$
4. \[ \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 0 \]

Find \( u(x, y) \) such that

5. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (Laplace’s eqn.)} \]
TYPE:

If the unknown function depends on a single independent variable, then the equation is an **ordinary differential equation (ODE)**.

If the unknown function depends on more than one independent variable, then the equation is a **partial differential equation (PDE)**.
ORDER:

The order of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.
Examples:

1. \( y' = 2x + \cos x \)

2. \( \frac{dy}{dx} = ky \) (exponential growth/decay)

3. \( x^2y'' - 2xy' + 2y = 4x^3 \)
4. \[ \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 0 \]

5. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (Laplace's eqn.)} \]

6. \[ \frac{d^2 y}{dx^2} + 2x \sin \left( \frac{dy}{dx} \right) + 3e^{xy} = \frac{d^3}{dx^3} (e^{2x}) \]
2. SOLUTIONS OF DIFFERENTIAL EQUATIONS

A solution of a differential equation is a function defined on some domain $D$ such that the equation reduces to an identity when the function is substituted into the equation.
Examples:

1. \( y' = 2x + \cos x \)
2. \( y' = ky \)
3. \( y'' - 2y' - 8y = 4e^{2x} \)

Is \( y = 2e^{4x} - \frac{1}{2}e^{2x} \) a solution?
\[ y'' - 2y' - 8y = 4e^{2x} \]

Is \( y = e^{-2x} + 2e^{3x} \) a solution?
4. \[ x^2 y'' - 4xy' + 6y = 3x^4 \]

Is \[ y = \frac{3}{2} x^4 + 2x^3 \] a solution?
\[ x^2 y'' - 4xy' + 6y = 3x^4 \]

Is \[ y = 2x^2 + x^3 \] a solution?
5. \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)

\[ u = \ln \sqrt{x^2 + y^2} \]  Solution?
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u = \cos x \sinh y, \quad u = 3x - 4y \]

Solutions??
3. Finding solutions - simple equations (from calculus)

1. Find the solutions of

\[ y' = 6x^2 + 4 \cos 2x \]

2. Find the solutions of

\[ y'' = 6e^{3x} + 12x \]
# Finding solutions - with hints

1. Find a value of $r$, if possible, such that $y = e^{rx}$ is a solution of

$$y'' - 3y' - 10y = 0.$$
2. Find a value of $r$, if possible, such that $y = x^r$ is a solution of

$$x^2y'' + 2xy' - 6y = 0.$$
3. Find a value of \( r \), if possible, such that \( y = x^r \) is a solution of

\[
y'' - \frac{1}{x} y' - \frac{3}{x^2} y = 0.
\]
Example: Find solutions of the differential equation:

\[ y''' - 12x + 6e^{2x} = 0 \]
NOTE: To solve a differential equation having the special form

\[ y^{(n)}(x) = f(x), \]

simply integrate \( f \) \( n \) times,

and EACH integration step produces an arbitrary constant;

there will be \( n \) independent arbitrary constants.
Intuitively, to find a set of solutions of an \( n \)-th order differential equation

\[
F \left[ x, y, y', y'', \ldots, y^{(n)} \right] = 0
\]

we “integrate” \( n \) times, with each integration step producing an arbitrary constant of integration (i.e., a parameter). Thus, ”in theory,” an \( n \)-th order differential equation has an \( n \)-parameter family of solutions.
SOLVING A DIFFERENTIAL EQUATION:

To solve an $n$-th order differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

means to find an $n$-parameter family of solutions. (Note: Same $n$.)

NOTE: An “$n$-parameter family of solutions” is more commonly called the GENERAL SOLUTION.
Examples: Find the general solution:

1. \( y' - 3x^2 - 2x + 4 = 0 \)
\[ y = x^3 + x^2 - 4x + C \]
2. \( y'' + 2 \sin 2x = 0 \)
\[ y = \frac{1}{2} \sin 2x + C_1 x + C_2 \]
3. \[ y''' - 3y'' + 3y' - y = 0 \]

**Answer:** \[ y = C_1 e^x + C_2 xe^x + C_3 x^2 e^x \]

4. \[ x^2 y'' - 4xy' + 6y = 3x^4 \]

**Answer:** \[ y = C_1 x^2 + C_2 x^3 + \frac{3}{2} x^4 \]
PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a particular solution.
Examples:

1. \( y'' = 6x + 8e^{2x} \)

General solution:

\[ y = x^3 + 2e^{2x} + C_1x + C_2 \]

Particular solutions:
2. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

General solution:

\[ y = C_1 x + C_2 x^2 + 2x^3 \]

Particular solutions:
THE DIFFERENTIAL EQUATION OF AN $n$-PARAMETER FAMILY:

Given an $n$-parameter family of curves.

The differential equation of the family is an $n$-th order differential equation that has the given family as its general solution.
Examples:

1. \( y^2 = Cx^3 + 4 \) is the general solution of a DE.

   a. What is the order of the DE?

   b. Find the DE?
2. \( y = C_1 x + C_2 x^3 \) is the general solution of a DE.

a. What is the order of the DE?

b. Find the DE?
General strategy for finding the differential equation of an n-parameter family

Step 1. Differentiate the family \( n \) times. This produces a system of \( n + 1 \) equations.

Step 2. Choose any \( n \) of the equations and solve for the parameters.

Step 3. Substitute the “values” for the parameters in the remaining equation.
Examples:

The given family of functions is the general solution of a differential equation.

(a) What is the order of the equation?

(b) Find the equation.
1. \[ y = Cx^3 - 2x \]

(a)

(b)
2. \[ y = C_1 e^{2x} + C_2 e^{3x} \]

(a) 

(b)
3. \[ y = C_1 \cos 3x + C_2 \sin 3x \]

(a)

(b)
4. \[ y = C_1 x^4 + C_2 x + C_3 \]

(a)

(b)
5. \[ y = C_1 + C_2x + C_3x^2 \]

(a)

(b)
INITIAL-VALUE PROBLEMS: (Text, Section 1.4)

1. Find a solution of

\[ y' = 3x^2 + 2x + 1 \]

which passes through the point \((-2, 4)\); that is, satisfies \(y(-2) = 4\).
\[ y = x^3 + x^2 + x + C \] (the general solution)
\[ y = x^3 + x^2 + x + 10 \] (the particular solution that satisfies the equation)
2. \( y = C_1 \cos 3x + C_2 \sin 3x \) is the general solution of

\[ y'' + 9y = 0. \]

a. Find a solution which satisfies

\[ y(0) = 3 \]
b. Find a solution which satisfies

\[ y(0) = 3, \quad y'(0) = 4 \]
\[ y = 3 \cos 3x + \frac{4}{3} \sin 3x \] is the solution of

\[ y'' + 9y = 0, \quad y(0) = 3, \quad y'(0) = 4. \]
c. Find a solution which satisfies

\[ y(0) = 4, \quad y(\pi) = 4 \]

d. Find a solution which satisfies

\[ y(0) = 4, \quad y(\pi) = -4 \]
An \emph{n-th order initial-value problem} consists of an \emph{n-th order differential equation}

\[ F\left[ x, y, y', y'', \ldots, y^{(n)} \right] = 0 \]

together with \emph{n} (initial) conditions of the form

\[ y(c) = k_0, \ y'(c) = k_1, \ y''(c) = k_2, \ldots, \]

\[ y^{(n-1)}(c) = k_{n-1} \]

where \( c \) and \( k_0, k_1, \ldots, k_{n-1} \) are given numbers.
1. An $n$-th order differential equation can always be written in the form

$$ F[x, y, y', y'', \ldots, y^{(n)}] = 0 $$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.
Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.
Examples:

1. Find a solution of the initial-value problem

\[ y' = 4x + 6e^{2x}, \quad y(0) = 5 \]
General solution: \( y = 2x^2 + 3e^{2x} + C \)
2. \( y = C_1 e^{-2x} + C_2 e^{4x} \) is the general solution of

\[ y'' - 2y' - 8y = 0 \]

Find a solution that satisfies the initial conditions

\[ y(0) = 3, \quad y'(0) = 2 \]
EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem have a solution? That is, do solutions to the problem exist?

2. If a solution does exist, is it unique? That is, is there exactly one solution to the problem or is there more than one solution?
3. \( y = C_1 x + C_2 x^3 \) is the general solution of

\[
y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0
\]

a. Find a solution which satisfies

\[
y(1) = 2, \quad y'(1) = -2.
\]
Graph: $y = 2x - 2x^3$
b. Find a solution which satisfies

\[ y(0) = 0, \quad y'(0) = 2. \]
Graphs: \( y = 2x + C_2x^3 \)
c. Find a solution which satisfies
\[ y(0) = 2, \quad y'(0) = -2. \]
Chapter 1. Terms

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