Chapter 2, Part 1

FIRST ORDER EQUATIONS

\[ F(x, y, y') = 0 \]

Basic assumption: The equation can be solved for \( y' \); that is, the equation can be written in the form

\[ y' = f(x, y) \]  \hspace{1cm} (1) \]
Assumed Background Material:

Techniques of integration, including:

- Substitution (the most common technique)
- Integration-by-parts
- Integrals involving trig functions
- Partial fraction decomposition
2.1. First Order Linear Equations

Equation (1) is a linear equation if $f$ has the form

$$f(x, y) = P(x)y + q(x)$$

where $P$ and $q$ are continuous functions on some interval $I$. Thus

$$y' = P(x)y + q(x)$$
Standard form:

The standard form for a first order linear equation is:

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on the interval \( I \)

(Note: A differential equation which is not linear is called nonlinear.)
Examples:

1. Find the general solution:

\[ y' = 3y \]
2. Find the general solution:

\[ y' + 2xy = 4x \]
Solution Method:

Step 1. Determine that the equation is linear and write it in standard form

\[ y' + p(x)y = q(x). \]
\[ y' + p(x)y = q(x). \]

**Step 2.** Multiply by \( e^{\int p(x) \, dx} : \)
\[
\left[ e^{\int p(x) \, dx} y \right]' = q(x) e^{\int p(x) \, dx}
\]

**Step 3.** Integrate:

\[
e^{\int p(x) \, dx} y = \int q(x) e^{\int p(x) \, dx} \, dx + C.
\]

**Step 4.** Solve for \( y \) :

\[
y = e^{-\int p(x) \, dx} \int q(t) e^{\int p(t) \, dt} \, dt + C e^{-\int p(x) \, dx}.
\]
\[ y = e^{-\int p(x) \, dx} \left\{ \int q(x) e^{\int p(x) \, dx} \, dx \right\} + Ce^{\int p(x) \, dx}. \]

is the general solution of the equation.

**Note:** \( e^{\int p(x) \, dx} \) is called an integrating factor
3. Find the general solution:

\[ xy' = \frac{\ln x}{x^2} - 3y \]
4. Find the general solution:

\[ xy' = \frac{2}{\sqrt{x^2 - 1}} - 2y + 2 \]
5. Solve the initial-value problem:

\[ y' + \cot x \cdot y = 2 \cos x, \quad y(\pi/2) = 3 \]
6. Find the general solution:

\[ y' + 2xy = 2 \tan x \]

Answer:

\[ y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2} \]
Answers:

1. \( y = C e^{3x} \)

2. \( y = 2 + C e^{-x^2} \)

3. \( y = \ln \frac{x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3} \)

4. \( y = \frac{2\sqrt{x^2 - 1}}{x^2} + 1 + \frac{C}{x^2} \)

5. \( y = \frac{5 - \cos 2x}{2 \sin x} \)
The term “linear:”

Differentiation:

As you know: for differentiable functions $f$ and $g$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

and for any constant $c$

$$\frac{d}{dx}[c f(x)] = c \frac{df}{dx}$$
Integration:

For integrable functions $f$ and $g$:

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

and, for any constant $c$

$$\int c f(x) \, dx = c \int f(x) \, dx$$
Any “operation” $L$ which satisfies

$$L[f(x) + g(x)] = L[f(x)] + L[g(x)]$$

and

$$L[c \, f(x)] = c \, L[f(x)]$$

is a “linear” operation.

1. **Differentiation** is a linear operation.

2. **Integration** is a linear operation.
Set \( L[y] = y' + p(x)y \)

\[
L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)
\]

\[
= y_1' + y_2' + py_1 + py_2
\]

\[
= y_1' + py_1 + y_2' + py_2 = L[y_1] + L[y_2]
\]

\[
L[cy] = (cy)' + p(cy) = cy' + cpy
\]

\[
= c(y' + py) = cL[y]
\]
Thus, if \( L[y] = y' + p(x)y \), then

\[
L[y_1 + y_2] = L[y_1] + L[y_2]
\]

\[
L[c \, y] = c \, L[y]
\]

\( L[y] = y' + p(x)y \) is a linear operation; \( L \) is a **linear operator**.

Hence the term linear differential equation.
Example:

\[ L[y] = y' + \frac{2}{x} y \]

Set \( y = x^2 \)

\[ L[x^2] = \]

Set \( y = x^3 + x \)

\[ L[x^3 + x] = \]
2.2. Separable Equations

\[ y' = f(x, y) \]

is a **separable equation** if \( f \) has the **factored form**

\[ f(x, y) = p(x)h(y) \]

where \( p \) and \( h \) are continuous functions. Thus

\[ y' = p(x)h(y) \]

is the "standard form" of a separable equation.
Example: Find the general solution of:

\[ y' = 2x + 2xy^2 \]
Solution Method

**Step 1.** Establish that the equation is separable.

**Step 2.** Divide both sides by \( h(y) \) to “separate” the variables.

\[
\frac{1}{h(y)} y' = p(x) \quad \text{or} \quad q(y)y' = p(x)
\]

which, in differential form, is:

\[
q(y) \, dy = p(x) \, dx.
\]

the variables are “separated.”
Step 3. Integrate

\[ \int q(y) \, dy = \int p(x) \, dx + C \]

\[ Q(y) = P(x) + C \]

where \( Q'(y) = q(y), \ P'(x) = p(x) \)
Note:

\[ Q(y) = P(x) + C \] is the general solution. Typically, this is an implicit relation; you may or may not be able to solve it for \( y \).
Examples:

1. Find the general solution:

\[ y' = \frac{xy^2 + 4x}{2y} \]
2. Find the general solution:

\[
\frac{dy}{dx} = 4x\sqrt{y - 2}
\]
Singular Solutions

\[ \frac{dy}{dx} = 4x\sqrt{y - 2} \]
3. Find the general solution:

\[ \frac{dy}{dx} - xy^2 = -x \]
4. Find the general solution:

\[
\frac{dy}{dx} = \frac{e^{x-y}}{1 + e^x}
\]
5. The equation

\[ y' = x(y + 2) \quad \text{or} \quad y' - xy = 2x \]

is both linear and separable. Find the general solution both ways.
Answers

1. \( y = \tan(x^2 + C) \)

2. \( \sqrt{y - 2} = x^2 + C \)

3. \( y = \frac{1 + C e^{x^2}}{1 - C e^{x^2}} \)

4. \( y = \ln \left[ \ln(1 + e^x) + C \right] \)

5. \( y = C e^{x^2/2} - 2 \)
2.3. Related Equations & Transformations

A. Bernoulli equations

An equation of the form

\[ y' + p(x)y = q(x)y^k, \quad k \neq 0, 1 \]

is called a Bernoulli equation.
The change of variable

\[ v = y^{1-k} \]

transforms a Bernoulli equation into

\[ v' + (1 - k)p(x)v = (1 - k)q(x). \]

which has the form

\[ v' + P(x)v = Q(x), \]

a linear equation.
Examples:

1. Find the general solution:

\[ y' - 4y = 2e^x \sqrt{y} \]
2. Find the general solution:

\[ xy' + y = 3x^3 y^2 \]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]
Answers

1. \[ y = (Ce^{2x} - e^x)^2 \]

2. \[ y = \frac{2}{Cx - 3x^3} \]

3. \[ y^2 = Cx^4 - x^2 \]
B. Homogeneous equations

\[ y' = f(x, y) \quad (1) \]

is a **homogeneous equation** if

\[ f(tx, ty) = f(x, y) \]
If (1) is homogeneous, then the change of dependent variable

\[ y = vx, \quad y' = v + xv' \]

transforms (1) into a separable equation:

\[ y' = f(x, y) \rightarrow v + xv' = f(x, vx) = f(1, v) \]

which can be written

\[ \frac{1}{f(1, v) - v} \, dv = \frac{1}{x} \, dx; \]

the variables are separated.
Examples:

1. Find the general solution:

\[ y' = \frac{x^2 + y^2}{2xy} \]
2. Find the general solution:

\[
\frac{dy}{dx} = \frac{x^2 e^{y/x} + y^2}{xy}
\]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]
Answers

1. \[ y^2 = x^2 - Cx \]

2. \[ y + x = e^{y/x} \left[ Cx - x \ln x \right] \]

3. \[ y^2 = Cx^4 - x^2 \]