EMCF #1.

Pb 2.

\[ e^\int_{\frac{3}{x}}^x \frac{3}{x} \, dx = e^{3 \ln x + C} \]

\[ = e^{\frac{3}{x} \ln x + C} \]

\[ = e^{3 \ln x + C} \cdot e \]

\[ = e^{3 \ln x} \cdot e \]

\[ = e^{\ln x^3} \cdot e \]

\[ = C x^3 \]

\[ a \ln x = \ln x^a \]
Chapter 2, Part 1

FIRST ORDER EQUATIONS

\[ F(x, y, y') = 0 \]

**Basic assumption:** The equation can be solved for \( y' \); that is, the equation can be written in the form

\[ y' = f(x, y) \quad (1) \]
Assumed Background Material:

Techniques of integration, including:

- Substitution (the most common technique)
- Integration-by-parts
- Integrals involving trig functions
- Partial fraction decomposition
2.1. First Order Linear Equations

\[ y' = f(x, y) \]

Equation (1) is a **linear equation** if \( f \) has the form

\[ f(x, y) = P(x)y + q(x) \]

where \( P \) and \( q \) are continuous functions on some interval \( I \). Thus

\[ y' = P(x)y + q(x) \]

\[ y' - P(x)y = q(x) \]

\[ y' + p(x)y = g(x) \]
Standard form:

The standard form for a first order linear equation is:

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on the interval \( I \)

(Note: A differential equation which is not linear is called nonlinear.)
Examples:

1. Find the general solution:

\[ y' = 3y \]

\[ y' - 3y = 0 \]

\[ p(x) = -3 \quad q(x) = 0 \]

Multiply by \( e^{-3x} \)

\[ e^{-3x} y' - 3e^{-3x} y = 0 \]

\[ (e^{-3x} y)' = 0 \]

\[ e^{-3x} y = C \]

\[ y = C e^{3x} \]
2. Find the general solution:

\[ y' + 2xy = 4x \]

\[ y = 2 + Ce^{-x^2} \quad \text{gen soln} \]
\[ y' + 2xy < 4x \]

mult by \( e^{\int 2x \, dx} = e^{x^2} \)

\[ e^{x^2} y' + e^{x^2} 2x e^{x^2} y = e^{x^2} 4x \]

Conclusion: you don't need to include the arbitrary constant when you calculate \( e^{\int 2x \, dx} \)
Solution Method:

Step 1. Determine that the equation is linear and write it in standard form

\[ y' + p(x)y = q(x). \]
Step 2. Multiply by \( e^{\int p(x) \, dx} \):

\[
y' + p(x)y = q(x).
\]

\[
e^{\int p(x) \, dx} y' + p e^{\int p(x) \, dx} y = e^{\int p(x) \, dx} q(x)
\]

\[
(e^{\int p(x) \, dx} y)' = e^{\int p(x) \, dx} q(x)
\]

\[
c \cdot y = \int e^{\int p(x) \, dx} q(x) \, dx + C
\]

\[
y = e^{-\int p(x) \, dx} \left[ \int q(x) e^{\int p(x) \, dx} \, dx + C \right]
\]
\[ \int \left[ e^{\int p(x) \, dx} \, y \right]' = q(x) e^{\int p(x) \, dx} \]

**Step 3.** Integrate:

\[ e^{\int p(x) \, dx} \, y = \int q(x) e^{\int p(x) \, dx} \, dx + C. \]

**Step 4.** Solve for \( y \):

\[ y = e^{-\int p(x) \, dx} \int q(t) e^{\int p(t) \, dt} \, dx + Ce^{-\int p(x) \, dx}. \]
\[ y = e^{-\int p(x) \, dx} \int q(x) e^{\int p(x) \, dx} \, dx + Ce^{-\int p(x) \, dx}. \]

is the general solution of the equation.

Note: \( e^{\int p(x) \, dx} \) is called an integrating factor.
3. Find the general solution:

\[ xy' = \frac{\ln x}{x^2} - 3y \]

**Step 1:**

\[ xy' + 3y = \frac{\ln x}{x^2} \]

\[ y' + \frac{3}{x}y = \frac{\ln x}{x^3} \]

2) **mul by** \( e^{\int \frac{3}{x} \, dx} \)

\[ e^{\int \frac{3}{x} \, dx} = e^{3\ln x} = e^{\ln x^3} = x^3 \]

\[ x^3 y' + 3x^2 y = \ln x \]

\( (x^3 y)' = \ln x \)

\[ x^3 y = \int \ln x \, dx = x\ln x - x + C \]

\[ y = \frac{\ln x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3} \quad \text{Gen Soln} \]
4. Find the general solution:

\[ xy' = \frac{2}{\sqrt{x^2 - 1}} - 2y + 2 \]

\[ xy' + 2y = \frac{2}{\sqrt{x^2 - 1}} \]

1) \[ y' + \frac{2}{x} y = \frac{2}{x \sqrt{x^2 - 1}} + \frac{2}{x} \]

2) Multiply by \( e^{\int \frac{2}{x} dx} = e^{\ln x^2} = x^2 \)

3) \[ x^2 y' + 2x y = \frac{2x}{\sqrt{x^2 - 1}} + 2x \]

\[ (x^2 y)^{\frac{1}{2}} = \frac{2x}{\sqrt{x^2 - 1}} + 2x \]
\[ x^2 y = \int \frac{2x}{\sqrt{x^2 - 1}} + 2x \, dx \]

\[ = \int u^{-\frac{1}{2}} \, du + x^2 + C \]

\[ = 2 \sqrt{x^2 - 1} + x^2 + C \]

\[ y \leq 2 \sqrt{\frac{x^2 - 1}{x^2}} + 1 + \frac{C}{x^2} \]

Gen soln
5. Solve the initial-value problem:

\[ y' + (\cot x)y = 2 \cos x, \quad y(\pi/2) = 3 \]

Already in std. form

\[ \begin{align*}
\text{mult by } e^{\int \cot x \, dx} &= e^{\ln \sin x} \\
&= \sin x
\end{align*} \]

\[(\sin x)y' + (\cos x)y = 2 \sin x \cos x \]

\[ (\sin x)y = \int 2 \cos 2x \, dx \]

\[ = -\frac{1}{2} \cos 2x + C \]

\[ y = -\frac{1}{2} \frac{\cos 2x}{\sin x} + \frac{C}{\sin x} \quad \text{gen soln} \]
\[ y' + \cot x \cdot y = 2 \cos x, \quad y\left(\frac{\pi}{2}\right) = 3 \]

\[ y = -\frac{\cos 2x}{2 \sin x} + \frac{C}{\sin x} \]

\[ y\left(\frac{\pi}{2}\right) = -\frac{\cos \left(\frac{\pi}{2}\right)}{2 \sin \left(\frac{\pi}{2}\right)} + \frac{C}{\sin \left(\frac{\pi}{2}\right)} \]

\[ \frac{1}{2} + C = 3 \]

\[ C = \frac{5}{2} \]

\[ y = -\frac{\cos 2x}{2 \sin x} + \frac{5}{2 \sin x} \]

\[ y = \frac{5 - \cos 2x}{2 \sin x} \]

Particular Solution

Solution of the initial-value problem.
\[ (\sin x \cdot y)' + \cos x \cdot y = 2 \sin x \cos x \]

\[ (\sin x \cdot y) = \int 2 \sin x \cos x \, dx \]

\[ u = \sin x \]

\[ \int 2u \, du \]

\[ \sin x \cdot y = \sin^2 x + C \]

\[ y = \frac{\sin x + \frac{2}{\sin x}}{\sin x} \]

\[ \cos 2x = 1 - 2\sin^2 x \]

\[ y(\frac{\pi}{2}) = 3 = 1 + C, \quad C = 2 \]

Same as above? Check it out.
6. Find the general solution:

\[ y' + 2xy = 2\tan x \]

\[ e^{x^2} y' + 2x e^{x^2} y = 2e^{x^2} \tan x \]

\[ (e^{x^2} y)' = \int 2e^{x^2} \tan x \, dx \]

\[ e^{x^2} y = \text{cannot calculate this integral} \]

Answer:

\[ y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2} \]
Answers:

1. \( y = Ce^{3x} \)

2. \( y = 2 + Ce^{-x^2} \)

3. \( y = \frac{\ln x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3} \)

4. \( y = \frac{2\sqrt{x^2 - 1}}{x^2} + 1 + \frac{C}{x^2} \)

5. \( y = \frac{5 - \cos 2x}{2\sin x} = \sin x + \frac{2}{\sin x} = \frac{2 + \sin^2 x}{\sin x} = \frac{1}{\sin x} \)
The term “linear:”

Differentiation:

As you know: for differentiable functions $f$ and $g$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

and for any constant $c$

$$\frac{d}{dx}[c f(x)] = c \frac{df}{dx}$$
Integration:

For integrable functions $f$ and $g$:

\[ \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \]

and, for any constant $c$

\[ \int c f(x) \, dx = c \int f(x) \, dx \]
Any “operation” $L$ which satisfies

$$L [f(x) + g(x)] = L[f(x)] + L[g(x)]$$

and

$$L [c f(x)] = c L [f(x)]$$

is a “linear” operation.

1. **Differentiation** is a linear operation.

2. **Integration** is a linear operation.
Set \( L[y] = y' + p(x)y \)

\[
L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)
\]

\[
= y_1' + y_2' + py_1 + py_2
\]

\[
= y_1' + py_1 + y_2' + py_2 = L[y_1] + L[y_2]
\]

\[
L[cy] = (cy)' + p(cy) = cy' + cpy
\]

\[
= c(y' + py) = cL[y]
\]
Thus, if \( L[y] = y' + p(x)y \), then

\[
L[y_1 + y_2] = L[y_1] + L[y_2]
\]

\[
L[c \, y] = c \, L[y]
\]

\( L[y] = y' + p(x)y \) is a linear operation; \( L \) is a linear operator.

Hence the term linear differential equation.
Example:

\[ L[y] = y' + \frac{2}{x} y \]

Set \( y = x^2 \)

\[ L[x^2] = \left( x^2 \right)' + \left( \frac{2}{x} \right) x^2 \]
\[ = 2x + \frac{2}{x} x^2 = 4x \]

Set \( y = x^3 + x \)

\[ L[x^3 + x] = \left( x^3 + x \right)' + \frac{2}{x} (x^3 + x) \]
\[ = 3x^2 + 1 + 2x^2 + 2 \]
\[ = 5x^2 + 3 \]
2.2. Separable Equations

\[ y' = f(x, y) \]

is a **separable equation** if \( f \) has the **factored form**

\[ f(x, y) = p(x)h(y) \]

where \( p \) and \( h \) are continuous functions. Thus

\[ y' = p(x)h(y) \]

is the ”standard form” of a separable equation.
Example: Find the general solution of:

\[ y' = 2x + 2xy^2 \]

would not be linear separable.

\[ \frac{dy}{dx} = y' = 2x(1+y^2) \]

\[ \frac{1}{1+y^2} \, dy = 2x \, dx \]

\[ \int \frac{1}{1+y^2} \, dy = \int 2x \, dx \]

\[ \tan^{-1} y = x^2 + C \quad \text{gen soln} \]

\[ y = \tan (x^2 + C) \quad \text{gen soln} \]
Solution Method

Step 1. Establish that the equation is separable.

\[ y' = p(x) h(y) \]

Step 2. Divide both sides by \( h(y) \) to “separate” the variables.

\[ \frac{1}{h(y)} y' = p(x) \quad \text{or} \quad q(y) y' = p(x) \]

which, in differential form, is:

\[ q(y) \, dy = p(x) \, dx. \]

the variables are “separated.”
Step 3. Integrate

\[ \int q(y) \, dy = \int p(x) \, dx + C \]

\[ Q(y) = P(x) + C \]

where \( Q'(y) = q(y), \ P'(x) = p(x) \)

A form of the general solution - you may be able to simplify this.
Note:

\[ Q(y) = P(x) + C \] is the general solution. Typically, this is an implicit relation; you may or may not be able to solve it for \( y \).

In the case of 1st order linear:

\[ y = e^{-\int p \, dx} \left[ \int q e^{\int p \, dx} + C \right] \text{ soln.} \]

is an explicit relation.
Examples:

1. Find the general solution:

\[
\frac{dy}{dx} = y' = \frac{xy^2 + 4x}{2y} = \frac{x(y^2 + 4)}{2y}
\]

\[
\int \frac{2y}{y^2 + 4} \, dy = \int x \, dx
\]

\[
\frac{1}{u} \, du = \ln |y^2 + 4| = \frac{1}{2} x^2 + C
\]

\[
e^{\ln |y^2 + 4|} = e^{\frac{1}{2} x^2 + C}
\]

\[
y^2 + 4 = e^{\frac{1}{2} x^2} \cdot e^C = Ce^{\frac{1}{2} x^2}
\]

\[
y^2 = Ce^{\frac{1}{2} x^2} - 4
\]

\[
y^2 = Ce^{\frac{1}{2} x^2} - 4
\]

\[
y = \sqrt{Ce^{\frac{1}{2} x^2} - 4}
\]
$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

the danger of taking square root — you may lose solutions
2. Find the general solution:

\[
\frac{dy}{dx} = 4x\sqrt{y-2}
\]

divide by \(\sqrt{y-2}\)

\[
\int \frac{dy}{\sqrt{y-2}} = \int 4x \, dx
\]

\[
y = 2
\]

\[
y = 0
\]

?  

\[
0 = 4x \sqrt{2-2} = 4x \cdot 0 = 0
\]

\[
y = 2 \text{ is a soln}
\]

\[
\frac{1}{\sqrt{y-2}} \, dy = 4x \, dx
\]

\[
2\sqrt{y-2} = 2x^2 + C
\]
$\sqrt{y-2} = x^2 + C$  Gen Soln

$y = 2$ is a solution not included in the gen soln. There is no value of $C \Rightarrow y = 2$

Singular solution (see chap 1)
Singular Solutions

\[
\frac{dy}{dx} = 4x\sqrt{y - 2}
\]
$\sqrt{\gamma - 2} = 2 \gamma + 1$
3. Find the general solution:

\[
\frac{dy}{dx} - xy^2 = -x
\]

\[
\frac{dy}{dx} = xy^2 - x = x(y^2 - 1)
\]

Note: \( y \neq 1, -1 \)

\[
\int \frac{dy}{y^2 - 1} = \int x \, dx
\]

\[
\frac{1}{2} \ln(y - 1) - \frac{1}{2} \ln(y + 1) = \frac{1}{2} x^2 + C
\]

\[
\ln \left( \frac{y - 1}{y + 1} \right) = x^2 + C
\]

A form of the general solution:

\( y = -1 \), \( y = 1 \)
\[ \ln \left( \frac{y-1}{y+1} \right) = x^2 + C \]

Take exp both sides:

\[ \frac{y-1}{y+1} = e^{x^2+C} = e^C \cdot e^{x^2} = C e^{x^2} \]

\[ y-1 = C e^{x^2} \quad \text{another form of general solution} \]

\( C = 0 \Rightarrow y = 1 \) is not a solution.

You can solve this for \( y \):

\[ y = \frac{1 + C e^{x^2}}{1 - C e^{x^2}} \]

Note \( C = 0 \) \( \Rightarrow y = 1 \)

\( y = -1 \) is not a solution.
4. Find the general solution:

\[
\frac{dy}{dx} = \frac{e^x - y}{1 + e^x} = \frac{e^x - y}{1 + e^x}
\]

\[
\int e^y \, dy = \int \frac{e^x}{1 + e^x} \, dx
\]

\[
e^y = \ln(1 + e^x) + C
\]

\[
\frac{dy}{dx} = \frac{e^x - y}{1 + e^x}
\]

\[
t = 1 + e^x
\]

\[
du = e^x \, dx
\]

\[
e^y = \ln(1 + e^x) + C
\]

\[
\text{gen soln}
\]

\[
y = \ln \left[ \ln (1 + e^x) + C \right]
\]

\[
\text{gen soln}
\]

\[
y = \ln \left[ \ln C (1 + e^x) \right]
\]

\[
\ln(1 + e^x) + \ln C
\]
5. The equation

\[ y' = x(y+2) \quad \text{or} \quad y' - xy = 2x \]

is both linear and separable. Find the general solution both ways.

\[ \ln(y+2) = \frac{1}{2}x^2 + C \]

\[ y+2 = e^{\frac{1}{2}x^2} + C \]

\[ y+2 = Ce^{\frac{1}{2}x^2} \]

\[ y=-2 \quad \text{not sing} \]

\[ C=0 \Rightarrow y=-2 \]
\[ y' = xy = 2x \]

div

must by \( e^{-x^2} \)

\[ e^{-x^2} y' - x e^{-x^2} y = 2x e^{-x^2} \]

\[ (e^{-x^2} y)' = 2x e^{-x^2} \]

separable to easier

and so on
Answers

1. \( y = \tan(x^2 + C) \)

2. \( \sqrt{y - 2} = x^2 + C \)

3. \( y = \frac{1 + Ce^{x^2}}{1 - Ce^{x^2}} \)

4. \( y = \ln \left[ \ln(1 + e^x) + C \right] \)
   
   (or \( \ln \ln [C (1 + e^x)] \))

5. \( y = Ce^{x^2}/2 - 2 \)
2.3. Related Equations & Transformations

A. Bernoulli equations

An equation of the form

\[ y' + p(x)y = q(x)y^k, \quad k \neq 0, 1 \]

is called a Bernoulli equation.

\[ y^{-k}y' + p y^{1-k} = q(x) \]

Divide by \( y^k \)

Mult by \( y^k \)

\( y^{1-k} \)

Set \( v = y^{1-k} \)

Change of variable

\( \text{screws up being linear} \)
The change of variable

\[ v = y^{1-k} \]
	ransforms a Bernoulli equation into

\[ v' + (1-k)p(x)v = (1-k)q(x). \]

which has the form

\[ v' + P(x)v = Q(x), \]

a linear equation.
Examples:

1. Find the general solution:

\[ y' - 4y = 2e^x \sqrt{y} \]

Divide by \( y^{\frac{1}{2}} \) (mult by \( y^{-\frac{1}{2}} \))

\[ y^{-\frac{1}{2}}y' - 4y^{\frac{1}{2}} = 2e^x \]

Set \( u = y^{\frac{1}{2}} \), \( v = \frac{1}{2} y^{-\frac{1}{2}} \)

\[ u' = 2v' \]

\[ 2v' - 4v = 2e^x \]
\[ v' - 2v = e^x \] linear

\[ \text{multi} \ e^{-2x} \ \text{to} \ e^{-2x} \]

\[ e^{-2x}v - 2e^{-2x}v = e^{-2x} \]

\[ e^{-2x}v = e^{-x} \]

\[ (e^{-x}v)' = e^{-x} \]

\[ e^{-2x}v = -e^{-x} + C \]

\[ v = Ce^{2x}e^{-x} \]

\[ \text{Recall} \ v = y \frac{1}{2} \]

\[ \text{we are} \ y' \text{to find} \ y \frac{1}{2} \]

\[ \text{Can't stop here} \text{trying} \]

\[ \text{So} \ y' = Ce^{2x}e^{-x} \]

\[ \text{get soln} \]
2. Find the general solution:

\[ xy' + y = 3x^3 y^2 \]

First divide eqn by \( x \):

\[ y' + \frac{1}{x} y = 3x^2 y^2 \]

Divide by \( y^2 \) (multi by \( y^{-2} \))

\[ y^{-2} y' + \frac{1}{x} y^{-1} = 3x^2 \]

Set \( v = y^{-1} \), \( v' = -y^{-2} y' \)

\[ y^{-2} y' = -v' \]
\[-V' + \frac{1}{x} \cdot V = 3x^2\]

\[\sqrt{-V} = -3x^2\] linear

multiply by \(e^{\int x \, dx} = e^{-\ln(x^3)} = x^{-3}\)

\[x^{-1} \cdot \frac{1}{x} \cdot V = -3x\]

\[(x^{-1}V)' = -3x\]

\[x^{-1}V = -\frac{3}{2}x^2 + C\]

\[V = Cx - \frac{3}{2}x^3\]

\[y^{-1} = Cx - \frac{3}{2}x^3 = Cx - \frac{3}{2}x^3\]

\[y = \frac{2}{Cx - \frac{3}{2}x^3}\]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]

\[ y' = \frac{x^2 + 2y^2}{xy} = \frac{1}{x} + \frac{2y}{x} \]

\[ y - \frac{2}{x} y = x \cdot y^{-1} \quad \text{Bernoulli} \]

Multiply by \( y \):

\[ yy' - \frac{2}{x} y^2 = x \]

\[ \sqrt{y}, \quad \sqrt{y} = 2yy' \quad yy' = \frac{1}{2} v' \]

\[ \frac{1}{2} v' = \frac{2}{x} \sqrt{v} \leq x \]
\[ v^{\prime} - \frac{v}{x} v = 2x \text{ linear} \]
mult by \[ e^{\int \frac{1}{x} dx} = e^{\ln x} = x \]
\[ x^{v^\prime} v - x v = 2x^{3} \]
\[ (x^{v^\prime}) = 2x^{3} \]
\[ x^{v^\prime} v = -x^{2} + C \]
\[ v = Cx^{u} - x^{2} \]
\[ y^{2} = Cx^{u} - x^{2} \text{ gen soln} \]
Answers

1. \[ y = (Ce^{2x} - e^x)^2 \]

2. \[ y = \frac{2}{Cx - 3x^3} \]

3. \[ y^2 = Cx^4 - x^2 \]
B. Homogeneous equations

\[ y' = f(x, y) \]  \hspace{1cm} (1)

is a \textbf{homogeneous equation} if

\[ f(tx, ty) = f(x, y) \]

\textbf{In general,}

if \( f \) \textbf{is homogeneous of degree} \( k \)

\[ f(tx, ty) = t^k f(x, y) \]
If (1) is homogeneous, then the change of dependent variable

\[ y = vx, \quad y' = v + xv' \]

transforms (1) into a separable equation:

\[ y' = f(x, y) \rightarrow v + xv' = f(x, vx) = f(1, v) \]

which can be written

\[ \frac{1}{f(1, v) - v} \, dv = \frac{1}{x} \, dx; \]

the variables are separated.
Examples:

1. Find the general solution:

\[ y' = \frac{x^2 + y^2}{2xy} \]

\[ f(tx, ty) < \frac{(tx)^2 + (ty)^2}{2 (tx)(ty)} \]

\[ = \frac{t^2 x^2 + t^2 y^2}{2t^2 xy} = \frac{x^2 + y^2}{2xy} \]

equ is homogeneous
Set \( y = vx \), \( y' = v + xv' \)

\[
y' = \frac{x^2 + y^2}{2xy}
\]

\[
\sqrt{1 + xv'} = \frac{x^2 + x^2v^2}{2x(1v)} = \frac{1 + v^2}{2v}
\]

\[
xv' = \frac{1 + v^2}{2v} - v = \frac{1 - v^2}{2v}
\]

\[
x \frac{dv}{dx} = \frac{1 - v^2}{2v}
\]

\[
\frac{2v}{1 - v^2} \, dv = \frac{1}{x} \, dx
\]
\[-\ln(1-v^2) = \ln x + C\]
\[\ln(1-v^2) = C - \ln x\]
\[1-v^2 = e^{C-\ln x} = e^C x\]
\[= \frac{C}{x}\]

\[v^2 = 1 - \frac{C}{x}\]
\[\frac{y^2}{x^2} = 1 - \frac{C}{x}, \quad y = x - Cx\]
2. Find the general solution:

\[
\frac{dy}{dx} = \frac{x^2 e^{y/x} + y^2}{xy}
\]

\[
f(tx, ty) = (tx)^{ty/tx} + (ty)^{ty/ty} = (tx)^{ty/tx + ty/ty}
\]

\[
= tx e^{ty/tx} + ty
\]

\[
= \frac{tx e^{ty/tx} + ty}{tx ty}
\]

\[
= \frac{x^2 e^{y/x} + y^2}{x y}
\]

homogeneous
Set \( y = vx \), \( y' = v + xv' \)

\[
v + xv' = \frac{e^v}{v} + xv'
\]

\[
x + xv' = e^v + \frac{v}{v}
\]

\[
x \frac{du}{dx} = \frac{e^v}{v} \quad \int \frac{dx}{c}
\]

\[
Ve^{-v} \int du = \frac{1}{x} \int dx
\]

\[
Ve^{-v} - e^{-v} = \ln x + C
\]

\[
V - \frac{1}{x} e^{-v} - e^{-v} = \ln x + C
\]

\[
V^{\frac{1}{2}} x - \frac{1}{x} e^{-v} - e^{-v} = \ln x + C
\]
\[
\frac{y}{x} e^{-y/2x} + e^{-y/2x} = C - \ln x
\]
\[
y e^{-y/2x} + x e^{-y/2x} = Cx - x \ln x
\]
\[
\frac{y}{e^{-y/2x}} + \frac{x}{e^{y/2x}} = Cx - x \ln x.
\]
\[
y + x = e^{y/2x} \left[ Cx - x \ln x \right]
\]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]

\[ y' = \frac{x^2 + 2y^2}{xy} \quad \text{homogeneous} \]

Set \( y = vx \), \( y' = v + xv' \)

\[ v + xv' = \frac{x^2 + 2xv^2}{x^2v} = \frac{1 + 2v^2}{v} \]

\[ xv' = \frac{1 + 2v^2}{v} - v = \frac{1 + v^2}{v} \]

\[ \frac{v}{1 + v^2} d\sqrt{\frac{1}{x}} = \frac{1}{x} dx \]

\[ \frac{1}{2} \ln(1 + v^2) = \ln x + C \]
\[ \ln (1 + v^2) = 2 \ln x + C \]
\[ \ln (1 + v^2) = \ln x^2 + C \]
\[ 1 + v^2 = e^{\ln x^2 + C} \]
\[ 1 + v^2 = e^{\ln x^2} \cdot e^C \]
\[ 1 + v^2 = C e^{\ln x^2} \]
\[ 1 + v^2 = C x^2 \]
\[ v = \sqrt{C x^2 - 1} \]
\[ y = C x - x^2 \]
Answers

1. \( y^2 = x^2 - Cx \)

2. \( y + x = e^{y/x} [Cx - x \ln x] \)

3. \( y^2 = Cx^4 - x^2 \)