EMCF #1.

Pb 2.

\[ e \int \frac{3}{x} \, dx = e^{3 \ln x + C} = e^{\ln x^3} = e^{\ln x^3} \cdot e^c = Cx^3 \]

\[ a) \ 3x \]
\[ b) x^3 \]
\[ c) e^{3x} \]
\[ d) \ln 3x \]

a \ln x = \ln x^a
FIRST ORDER EQUATIONS

\[ F(x, y, y') = 0 \]

**Basic assumption:** The equation can be solved for \( y' \); that is, the equation can be written in the form

\[ y' = f(x, y) \quad (1) \]
Assumed Background Material:

Techniques of integration, including:

- Substitution (the most common technique)
- Integration-by-parts
- Integrals involving trig functions
- Partial fraction decomposition
2.1. First Order Linear Equations

\[ y' = f(x, y) \]

Equation (1) is a **linear equation** if \( f \) has the form

\[ f(x, y) = P(x)y + q(x) \]

where \( P \) and \( q \) are continuous functions on some interval \( I \). Thus

\[ y' = P(x)y + q(x) \]

\[ y' - P(x)y = q(x) \]

\[ y' + p(x)y = q(x) \]
The **standard form** for a first order linear equation is:

$$ y' + p(x)y = q(x) $$

where $p$ and $q$ are continuous functions on the interval $I$

(Note: A differential equation which is not linear is called **nonlinear**.)
Examples:

1. Find the general solution:

\[ y' = 3y \]

\[ y' - 3y = 0 \]

\[ p(x) = -3 \quad q(x) = 0 \]

Multiply by \( e^{-3x} \)

\[ e^{-3x} y' - 3 e^{-3x} y = 0 \]

\[ (e^{-3x} y)' = 0 \]

\[ e^{-3x} y = C \]

\[ y = Ce^{3x} \]
2. Find the general solution:

\[ y' + 2xy = 4x \]

\[
\int 2x \, dx = e^{\int 2x \, dx} = e^{x^2}
\]

\[ p(x) = 2x \quad q(x) = 4x \]

\[ \text{mult by } e^{x^2} \]

\[
e^{x^2} y' + 2xe^{x^2} y = 4xe^{x^2}
\]

\[
(e^{x^2} y)' = 4xe^{x^2}
\]

\[
(e^{x^2} y) = \int 4xe^{x^2} \, dx = 2e^{x^2} + C
\]

\[ y = 2 + Ce^{-x^2} \quad \text{gen soln} \]
y' + 2x y = 4x

multiply by \( e^\int 2x \, dx \) = \( e^{x^2} \)

\( e^{x^2} y' + \int 2x \cdot e^{x^2} \, dx = \int 4x \cdot e^{x^2} \, dx \)

Conclusion: you don't need to include the arbitrary constant when you calculate \( e^{\int 2x \, dx} \)
Solution Method:

Step 1. Determine that the equation is linear and write it in standard form

\[ y' + p(x)y = q(x). \]
\[ y' + p(x)y = q(x). \]

**Step 2.** Multiply by \( e^{\int p(x) \, dx} \):

\[
e^{\int p(x) \, dx} y' + p(x) e^{\int p(x) \, dx} y = e^{\int p(x) \, dx} q(x)
\]

\[
\left( e^{\int p(x) \, dx} y \right)' = e^{\int p(x) \, dx} q(x)
\]

\[
C \cdot y = \int e^{\int p(x) \, dx} q(x) \, dx + C
\]

\[
y = e^{-\int p(x) \, dx} \left[ \int q(x) e^{\int p(x) \, dx} \, dx + C \right]
\]
\[ e^\int p(x) \, dx \ y' = q(x) e^\int p(x) \, dx \]

**Step 3.** Integrate:

\[ e^\int p(x) \, dx \ y = \int q(x) e^\int p(x) \, dx \, dx + C. \]

**Step 4.** Solve for \( y \):

\[ y = e^{-\int p(x) \, dx} \int q(t) e^\int p(t) \, dt \, dx + C e^{-\int p(x) \, dx}. \]
\[ y = e^{-\int p(x) \, dx} \left( \int q(x) e^{\int p(x) \, dx} \, dx \right) + Ce^{-\int p(x) \, dx}. \]

is the general solution of the equation.

**Note:** \( e^{\int p(x) \, dx} \) is called an **integrating factor**
3. Find the general solution:

\[ xy' = \frac{\ln x}{x^2} - 3y \]

\text{Step 1:} \quad xy' + 3y = \frac{\ln x}{x^2}
\[
y' + \frac{3}{x}y = \frac{\ln x}{x^3}
\]

2) \quad \text{Multiply by} \quad e^{\int \frac{3}{x} \, dx}
\[
\int e^x \, dx = e^x = e^{\ln x^3} = x^3
\]

\[ x^3y' + 3x^2y = \ln x \]
\[
(x^3 y)' = \ln x
\]
\[
x^3 y = \int \ln x \, dx = x \ln x - x + C
\]
\[
y = \frac{\ln x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3}
\]

Gen. Soln.
4. Find the general solution:

\[ xy' = \frac{2}{\sqrt{x^2 - 1}} - 2y + 2 \]

\[ xy' + 2y = \frac{2}{\sqrt{x^2 - 1}} + 2 \]

1) \[ y' + \frac{2}{x} y = \frac{2}{x \sqrt{x^2 - 1}} + \frac{2}{x} \]

2) Multiply by: \[ e^{\int \frac{2}{x} \, dx} = e^{\ln x^2} = x^2 \]

3) \[ x^2 y' + 2x y = \frac{2x}{\sqrt{x^2 - 1}} + 2x \]

\[ (x^2 y)' = \frac{2x}{\sqrt{x^2 - 1}} + 2x \]
\[ x^2 y = \int \left( \frac{2x}{\sqrt{x^2 - 1}} + 2x \right) \, dx \]

\[ = \int u^{-\frac{1}{2}} \, du + x^2 + C \]

\[ = 2 \sqrt{x^2 - 1} + x^2 + C \]

\[ y \leq 2 \sqrt{\frac{x^2 - 1}{x^2}} + 1 + \frac{C}{x^2} \]

Gen soln
5. Solve the initial-value problem:

\[ y' + (\cot x)y = 2\cos x, \quad y(\pi/2) = 3 \]

Already in std. form

multi by \[ e^{\int \cot x \, dx} = e^{\ln \sin x} = \sin x \]

\[ (\sin x)y' + (\cos x)y = 2\sin x \cos x \]

\[ [(\sin x)y]' = 2\sin x \cos x \]

\[ \sin x y = \int 2\sin 2x \, dx \]

\[ = -\frac{1}{2} \cos 2x + C \]

\[ y = -\frac{1}{2} \frac{\cos 2x}{\sin x} + \frac{C}{\sin x} \quad \text{gen soln} \]
\[ y' + \cot x \cdot y = 2 \cos x, \quad y\left(\frac{\pi}{2}\right) = 3 \]

\[ y = -\frac{\cos 2x}{2 \sin x} + \frac{C}{\sin x} \]

\[ y\left(\frac{\pi}{2}\right) = -\frac{\cos \left(\frac{\pi}{2}\right)}{2 \sin \left(\frac{\pi}{2}\right)} + \frac{C}{\sin \left(\frac{\pi}{2}\right)} \]

\[ = \frac{1}{2} + C = 3 \]

\[ C = \frac{5}{2} \]

\[ y = -\frac{\cos 2x}{2 \sin x} + \frac{5}{2 \sin x} \]

\[ = \frac{5 - \cos 2x}{2 \sin x} \]

particular solution

solution of the initial-value problem
\[(\sin x \cdot y)' + \cos x \cdot y = 2 \sin x \cos x\]

\[(\sin x \cdot y) = \int 2 \sin x \cos x \, dx\]

\[u = \sin x\]

\[\int 2u \, du\]

\[\sin x \cdot y = \sin^2 x + C\]

\[y = \sin x + \frac{2}{\sin x}\]

\[\cos 2x = 1 - 2\sin^2 x\]

\[y \left(\frac{\pi}{2}\right) = 3 = 1 + C, \quad C = 2\]

\[y = \sin x + \frac{2}{\sin x}\]

Same as \[\text{Solve above}\]:

Check it out
6. Find the general solution:

\[ y' + 2xy = 2 \tan x \]

\[ e^{\int 2x \, dx} e^{-x^2} = e^{-x^2} \int 2e^{x^2} \tan x \, dx \]

\[ (e^{x^2}y)' = \int 2e^{x^2} \tan x \, dx \]

\[ e^{x^2}y = \text{Cannot calculate this integral} \]

\[ e^{x^2} \tan x \text{ - no antiderivative} \]

Answer:

\[ y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2} \]
Answers:

1. \[ y = Ce^{3x} \]

2. \[ y = 2 + Ce^{-x^2} \]

3. \[ y = \frac{\ln x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3} \]

4. \[ y = \frac{2\sqrt{x^2 - 1}}{x^2} + 1 + \frac{C}{x^2} \]

5. \[ y = \frac{5 - \cos 2x}{2 \sin x} = \sin x + \frac{2}{\sin x} = \frac{2 + \sin^2 x}{\sin x} \]
The term “linear:”

Differentiation:

As you know: for differentiable functions $f$ and $g$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

and for any constant $c$

$$\frac{d}{dx}[c f(x)] = c \frac{df}{dx}$$
Integration:

For integrable functions $f$ and $g$:

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

and, for any constant $c$

$$\int c f(x) \, dx = c \int f(x) \, dx$$
Any “operation” $L$ which satisfies

$$L[f(x) + g(x)] = L[f(x)] + L[g(x)]$$

and

$$L[c f(x)] = c L[f(x)]$$

is a “linear” operation.

1. **Differentiation** is a linear operation.

2. **Integration** is a linear operation.
Set \( L[y] = y' + p(x)y \)

\[
L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)
\]

\[
= y'_1 + y'_2 + py_1 + py_2
\]

\[
= y'_1 + py_1 + y'_2 + py_2 = L[y_1] + L[y_2]
\]

\[
L[cy] = (cy)' + p(cy) = cy' + cpy
\]

\[
= c(y' + py) = cL[y]
\]
Thus, if \( L[y] = y' + p(x)y \), then

\[
L[y_1 + y_2] = L[y_1] + L[y_2]
\]

\[
L[c y] = c L[y]
\]

\( L[y] = y' + p(x)y \) is a linear operation; \( L \) is a **linear operator**.

Hence the term linear differential equation.
Example:

\[ L[y] = y' + \frac{2}{x} y \]

Set \( y = x^2 \)

\[ L[x^2] = (x^2)' + \left( \frac{2}{x} \right) x^2 = 2x + \frac{2}{x} \cdot x^2 = 4x \]

Set \( y = x^3 + x \)

\[ L[x^3 + x] = (x^3 + x)' + \frac{2}{x} (x^3 + x) = 3x^2 + 1 + 2x^2 + 2 = 5x^2 + 3 \]
2.2. Separable Equations

\[ y' = f(x, y) \]

is a **separable equation** if \( f \) has the **factored form**

\[ f(x, y) = p(x)h(y) \]

where \( p \) and \( h \) are continuous functions. Thus

\[ y' = p(x)h(y) \]

is the ”standard form” of a separable equation.
Example: Find the general solution of:

\[ y' = 2x + 2xy^2 \]

\[ \frac{dy}{dx} = y' = 2x(1+y^2) \]

Separable

\[ \frac{1}{1+y^2} \frac{dy}{dx} = 2x \]

\[ \int \frac{1}{1+y^2} \ dy = \int 2x \ dx \]

\[ \tan^{-1} y = x^2 + C \]

\[ \text{gen soln} \]

\[ y = \tan(x^2 + C) \]

\[ \text{gen soln}. \]
Solution Method

Step 1. Establish that the equation is separable.

\[ y' = p(x)h(y) \]

Step 2. Divide both sides by \( h(y) \) to "separate" the variables.

\[ \frac{1}{h(y)}y' = p(x) \quad \text{or} \quad q(y)y' = p(x) \]

which, in differential form, is:

\[ q(y)\,dy = p(x)\,dx. \]

the variables are "separated."
Step 3. Integrate

\[ \int q(y) \, dy = \int p(x) \, dx + C \]

\[ Q(y) = P(x) + C \]

where \( Q'(y) = q(y), \ P'(x) = p(x) \)

A form of the general solution - you may be able to simplify this.
Note:

\[ Q(y) = P(x) + C \] is the general solution. Typically, this is an implicit relation; you may or may not be able to solve it for \( y \).

In the case of 1st order linear

\[ y = e^{-\int P \, dx} \left[ \int q e^{\int P \, dx} + C \right] \]

this is an explicit relation.
Examples:

1. Find the general solution:

\[ \frac{dy}{dx} = y' = \frac{xy^2 + 4x}{2y} = \frac{x(y^2 + 4)}{2y} \]

\[ \int \frac{2y}{y^2 + 4} \, dy = \int x \, dx \]

Let \( u = y^2 + 4 \), then \( du = 2y \, dy \)

\[ \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln(u) + C \]

\[ \ln(y^2 + 4) = \frac{1}{2} x^2 + C \]

\[ e^{\ln(y^2 + 4)} = e^{\frac{1}{2} x^2 + C} \]

\[ y^2 + 4 = e^{\frac{1}{2} x^2} \cdot e^C \]

\[ y^2 + 4 = Ce^{\frac{1}{2} x^2} \]

\[ y^2 = Ce^{\frac{1}{2} x^2} - 4 \]

The simplified form of the general solution is \( y^2 = Ce^{\frac{1}{2} x^2} - 4 \).
\[ x^2 + y^2 = 1 \]

\[ y = \sqrt{1-x^2} \]

the danger of taking square root — you may lose solutions
2. Find the general solution:

\[ \frac{dy}{dx} = 4x \sqrt{y - 2} \]

\[ \int \frac{1}{\sqrt{y-2}} \, dy = \int 4x \, dx \]

\[ 2 \sqrt{y-2} = 2x + C \]
\sqrt{y-2} = x^2 + C \quad \text{Gen Soln}

y = 2 \text{ is a soln not included in the gen soln. There is no value of } C \Rightarrow y = 2

Singular solution (see chap 1)
Singular Solutions

\[ \frac{dy}{dx} = 4x\sqrt{y - 2} \]
3. Find the general solution:

\[ \frac{dy}{dx} - xy^2 = -x \]

\[ \frac{dy}{dx} = xy^{-2} - x = x(y^2 - 1) \]

Note: \( y \neq 1, -1 \)

\[ \int \frac{dy}{y^2 - 1} = \int x \, dx \]

\[ \frac{1}{2} \ln(y-1) - \frac{1}{2} \ln(y+1) = \frac{1}{2} x^2 + C \]

\[ \ln(y-1) - \ln(y+1) = x^2 + C \]

\[ \ln \left( \frac{y-1}{y+1} \right) = x^2 + C \]

A form of the general solution:

\( 1, -1 \) are singular solutions.
\[ \ln \left( \frac{y-1}{y+1} \right) = x^2 + C \]

Take exp both sides

\[ \frac{y-1}{y+1} = e^{x^2+C} = e^{x^2}e^C = Ce^{x^2} \]

\[ \frac{y-1}{y+1} = Ce^{x^2} \]

Another form of gen soln.

\[ C = 0 \Rightarrow y = 1, \text{ is not a soln} \]

\[ y = 1 \text{ is still a sing soln} \]

You can solve this for \( y \).

\[ y = \frac{1 + C e^{x^2}}{1 - C e^{x^2}} \]

Note \( C = 0 \Rightarrow y = 1 \)

\[ y = -1 \text{ is a sing soln.} \]
4. Find the general solution:

\[ \frac{dy}{dx} = \frac{e^{x-y}}{1 + e^x} = \frac{e^x e^{-y}}{1 + e^x} \]

\[ \int e^y dy = \int \frac{e^x}{1 + e^x} dx \]

\[ e^y = \ln(1 + e^x) + C \]

\[ y = \ln \left( \ln(1 + e^x) + C \right) \]

\[ \ln(1 + e^x) + \ln C \]
5. The equation

\[ y' = x(y + 2) \quad \text{or} \quad y' - xy = 2x \]

is both linear and separable. Find the general solution both ways.

\begin{align*}
\int dy &= x \, dx \\
y + 2 &= \frac{1}{2} x^2 + C \\
\ln(y + 2) &= \frac{1}{2} x^2 + C \\
y + 2 &= e^{\frac{1}{2} x^2 + C} \\
C &= 0 \Rightarrow y = -2
\end{align*}
\[ y' = xy = 2x \]

\[ \ln \text{ mult by } e \]

\[-x^2 y' - x e^{-x^2} y = 2x e^{-x^2} \]

\[ (e^{-x^2/2} y)' = 2x e^{-x^2/2} \]

(Seperable is easier)

and so on
Answers

1. \[ y = \tan(x^2 + C) \]

2. \[ \sqrt{y - 2} = x^2 + C \]

3. \[ y = \frac{1 + Ce^{x^2}}{1 - Ce^{x^2}} \]

4. \[ y = \ln \left[ \ln(1 + e^x) + C \right] \]
   or \[ \ln \left( \ln \left[ C (1 + e^x) \right] \right) \]

5. \[ y = Ce^{x^2/2} - 2 \]
2.3. Related Equations & Transformations

A. Bernoulli equations

An equation of the form

$$y' + p(x)y = q(x)y^k, \quad k \neq 0, 1$$

is called a Bernoulli equation.

Set $v = y^{-k}$

Change of variable

divide by

ie, mult by $y^{-k}$

screws up being linear
The change of variable

\[ v = y^{1-k} \]

transforms a Bernoulli equation into

\[ v' + (1 - k)p(x)v = (1 - k)q(x). \]

which has the form

\[ v' + P(x)v = Q(x), \]

a linear equation.
Examples:

1. Find the general solution:

\[ y' - 4y = 2e^x \sqrt{y} \]

Divide by \( \sqrt{y} \) (multiply by \( y^{-\frac{1}{2}} \))

\[ \frac{1}{2}y^{-\frac{1}{2}}y' - 4y^{\frac{1}{2}} = 2e^x \]

Set \( u = y^{\frac{1}{2}} \), \( u' = \frac{1}{2}y^{-\frac{1}{2}}y' \)

\[ y^{\frac{1}{2}}y' = 2u^2 \]

\[ 2u^2 - 4u = 2c \]
\[ v' - 2v = e^x \text{ linear} \]

multiply e^{-2x} to e^{-2x} \[ e^{-2x} v - 2e^{-2x} v = e^{-2x} \]

\[ (e^{-2x} v)' = e^{-x} \]

\[ e^{-2x} v = -e^{-x} + C \]

\[ v = Ce^{2x} - e^x \]

Can't stop here, trying...

we are trying to find \( y \frac{1}{2} \)

Recall \( v = y \)

So \( y \frac{1}{2} = Ce^{2x} - e^x \)

Gen soln...
2. Find the general solution:

\[ xy' + y = 3x^3 y^2 \]

Not linear

Bermoulli

\( n = 2 \)

First divide eqn by \( x \)

\[ y' + \frac{1}{x} y = 3x^2 y^2 \]

\[ y^{-2} y' + 1 \cdot \frac{1}{x} y^{-1} = 3x^2 \]

Set \( V = y^{-1} \)

\[ V = -y^{-2} y' \]

\[ y^{-2} y' = -V' \]
\[-V^1 + \frac{1}{X} \quad V = 3X^2\]
\[\sqrt{-V} = -3X^2 \quad \text{linear}\]
multiply \(e^{\int 1\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}\)

\[X^{-1} - x^{-2} V = -3X\]

\[(x^{-1} V)' = -3X\]

\[x^{-1} V = -\frac{3}{2}X + C\]

\[V = Cx - \frac{3}{2}x^3\]

\[y^{-1} = Cx - \frac{3}{2}x^3 = Cx -\frac{3}{2}x^3\]

\[y = \frac{2}{Cx^3/2} x^3\]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]

\[ y' = \frac{x^2 + 2y^2}{xy} = \frac{x}{y} \frac{1}{x} + \frac{2y}{x} \]

\[ y' - \frac{2}{x} y = x \cdot y^{-1} \text{ Bernoulli} \]

mult by \( y \)

\[ yyy' - \frac{2}{x} y^2 = x \]

\[ \sqrt{y}, \sqrt{y} \geq 2yy', \quad yy' = \frac{1}{2} v' \]

\[ \frac{1}{2} v^2 \geq \frac{2}{x} \sqrt{v} \leq x \]
\[ \sqrt{1 - \frac{y}{x}} = 2x \quad \text{linear} \]

mult by \( e^{\int \frac{y}{x} \, dx} = e^{\frac{-u}{x}} \)

\[ \frac{-u}{x} \cdot \frac{1}{x} = 2x^{-3} \]

\[ x^2 v - uvx^3 = 2x^{-3} \]

\[ (x^2 v) = 2x^{-3} \]

\[ x^2 v = -x^2 + C \]

\[ v = Cx - x \]

\[ \sqrt{y} = Cx - x \quad \text{gen soln} \]
Answers

1. \( y = (Ce^{2x} - e^x)^2 \)

2. \( y = \frac{2}{Cx - 3x^3} \)

3. \( y^2 = Cx^4 - x^2 \)
B. Homogeneous equations

\[ y' = f(x, y) \quad (1) \]

is a homogeneous equation if

\[ f(tx, ty) = f(x, y) \]

in general, \( f \) is homogeneous of degree \( k \) if

\[ f(tx, ty) = t^k f(x, y) \]
If (1) is homogeneous, then the change of dependent variable

\[ y = vx, \quad y' = v + xv' \]

transforms (1) into a separable equation:

\[ y' = f(x, y) \rightarrow v + xv' = f(x, vx) = f(1, v) \]

which can be written

\[ \frac{1}{f(1, v) - v} \, dv = \frac{1}{x} \, dx; \]

the variables are separated.
Examples:

1. Find the general solution:

\[ y' = \frac{x^2 + y^2}{2xy} \]

\[ f(tx, ty) = \frac{(tx)^2 + (ty)^2}{2(tx)(ty)} \]

\[ = \frac{t^2x^2 + t^2y^2}{2t^2xy} = \frac{x^2 + y^2}{2xy} \]

equ is homogeneous
Set \( y = v x \), \( y' = v + x v' \)

\[ y' = v^2 + y^2 \]

\[ y = \frac{v^2 + y^2}{2xy} \]

\[ \sqrt{1 + x v'} = \sqrt{y + x v'^2} = \frac{1 + v^2}{2v(vx)} \]

\[ x v' = \frac{1 + v^2}{2v} - v = \frac{1 - v^2}{2v} \]

\[ x \frac{dv}{dx} = \frac{1 - v^2}{2v} \]

\[ \frac{2v}{1 - v^2} dv = \frac{1}{x} dx \]
\[ -\ln(1-v^2) = \ln x + C \]
\[ \ln(1-v^2) = C - \ln x \]
\[ 1-v^2 = e^{C-\ln x} = e^C x^{-1} \]
\[ = \frac{C}{x} \]

\[ V^2 = 1 - \frac{C}{x} \]

\[ y = \frac{y^2}{x^2} = 1 - \frac{C}{x} \]
\[ y = \sqrt{x - Cx} \]
2. Find the general solution:

\[
\frac{dy}{dx} = \frac{x^2 e^{y/x} + y^2}{xy}
\]

\[
f(tx, ty) = (tx)^2 e^{ty/tx} + (ty)^2
\]

\[
= t^2 x^2 e^{y/x} + t^2 y^2
\]

\[
= \frac{t^2 x^2 e^{y/x} + t^2 y^2}{txy}
\]

\[
\rightarrow x^2 e^{y/x} + y^2
\]

\[
\rightarrow \frac{x^2 e^{y/x} + y^2}{xy}
\]

homogeneous
Set \( y = v^x \), \( y' = v + xv' \)

\[
v + xv' = v e^{\frac{vx}{v}} + x v
\]

\[
x + xv' \overset{e}{=\frac{e + v}{v}} = \frac{x + v}{v}
\]

\[
x \frac{du}{dx} = \frac{e^v}{v}
\]

\[
Ve^{-v} dv = \int \frac{dx}{x}
\]

\[
Ve^{-v} - e^{-v} = \ln x + C
\]

\[
\frac{y}{x} - \frac{y - \sqrt{x}}{x} e - e = \ln x + C
\]
\[ \frac{y}{x} e^{-\frac{y}{x}} + e^{\frac{y}{x}} = C - \ln x \]
\[ ye^{-\frac{y}{x}} + xe^{\frac{y}{x}} = Cx - x \ln x \]
\[ \frac{y}{e^{\frac{y}{x}}} + \frac{x}{e^{\frac{y}{x}}} = Cx - x \ln x. \]
\[ y + x = e^{\frac{y}{x}} \left[ Cx - x \ln x \right] \]
3. Find the general solution:

\[ xyy' = x^2 + 2y^2 \]
Answers

1. \( y^2 = x^2 - Cx \)

2. \( y + x = e^{y/x} \left[ Cx - x \ln x \right] \)

3. \( y^2 = Cx^4 - x^2 \)