Chapter 2, Part 1

FIRST ORDER EQUATIONS

\[ F(x, y, y') = 0 \]
Background Material: (Text, Section 2.1)

Techniques of integration:

• Substitution (the most common technique)

• Integration-by-parts

• Integrals involving trig functions

• Partial fraction decomposition
Given a first order DE

\[ F(x, y, y') = 0 \]

**Basic assumption:** The equation can be solved for \( y' \); that is, the equation can be written in the form

\[ y' = f(x, y) \quad (1) \]
2.2. FIRST ORDER LINEAR EQUATIONS

(y') = f(x, y)

is a linear equation if \( f \) has the form

\[ f(x, y) = P(x)y + q(x) \]

where \( P \) and \( q \) are continuous functions on some interval \( I \). Thus

\[ y' = P(x)y + q(x) \]
Standard form:

The **standard form** for a first order linear equation is:

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on the interval \( I \)

(Note: A differential equation which is not linear is called **nonlinear**.)
Examples:

1. Find the general solution:

\[ y' = ky, \ k \text{ constant} \quad (\text{See Examples in Chapter 1}) \]
2. Find the general solution:

\[ y' + 2xy = 4x \]
Solution Method:

Step 1. **Identify**: Determine that the equation IS linear and write it in standard form

\[ y' + p(x)y = q(x). \]
Step 2. Multiply by $e^\int p(x) \, dx$:
\[ \left[ e^{\int p(x) \, dx} \, y \right]' = q(x) e^{\int p(x) \, dx} \]

**Step 3.** Integrate:

\[ e^{\int p(x) \, dx} \, y = \int q(x) e^{\int p(x) \, dx} \, dx + C. \]

**Step 4.** Solve for \( y \):

\[ y = e^{-\int p(x) \, dx} \int q(t) e^{\int p(t) \, dt} \, dx + C e^{-\int p(x) \, dx}. \]
$$y = e^{-\int p(x) \, dx} \int q(x) e^{\int p(x) \, dx} \, dx + Ce^{-\int p(x) \, dx}$$

is the general solution of the equation.

**Note:** $e^{\int p(x) \, dx}$ is called an **integrating factor**
3. Find the general solution:

\[ xy' = \frac{\cos 2x}{x^2} - 3y \]
4. Find the general solution:

\[ xy' = \frac{2}{\sqrt{x^2 - 1}} - 2y + 2 \]
5. Solve the initial-value problem:

\[ y' + (\cot x)y = 2\cos x, \quad y(\pi/2) = 3 \]
6. Find the general solution:

\[ y' + 2xy = 2 \tan x \]
Answers:

1. $y = Ce^{kx}$

2. $y = 2 + Ce^{-x^2}$

3. $y = \frac{\sin 2x}{2x^3} + \frac{C}{x^3}$

4. $y = \frac{2\sqrt{x^2 - 1}}{x^2} + 1 + \frac{C}{x^2}$

5. $y = \frac{5 - \cos 2x}{2 \sin x}$

6. $y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2}$
Linear Operations and the term “linear”

Differentiation:

As you know: For differentiable functions $f$ and $g$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

and for any constant $c$

$$\frac{d}{dx}[c f(x)] = c \frac{df}{dx}$$
Integration:

For integrable functions $f$ and $g$:

\[
\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx
\]

and, for any constant $c$

\[
\int c f(x) \, dx = c \int f(x) \, dx
\]
Any “operation” $L$ which satisfies

$$L[f(x) + g(x)] = L[f(x)] + L[g(x)]$$

and

$$L[c f(x)] = c L[f(x)]$$

is a “linear” operation.

1. **Differentiation** is a linear operation.

2. **Integration** is a linear operation.
Set \( L[y] = y' + p(x)y \)

\[
L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)
\]

\[
= y_1' + y_2' + py_1 + py_2
\]

\[
= y_1' + py_1 + y_2' + py_2 = L[y_1] + L[y_2]
\]

\[
L[cy] = (cy)' + p(cy) = cy' + cpy
\]

\[
= c(y' + py) = cL[y]
\]
Thus, if \( L[y] = y' + p(x)y \), then

\[
L[y_1 + y_2] = L[y_1] + L[y_2]
\]

\[
L[c\,y] = c\,L[y]
\]

\( L[y] = y' + p(x)y \): the left-hand side of a linear differential equation in standard form; \( L \) is a **linear operator**. Hence the term linear differential equation.
Example. Let $L[y] = y' + \frac{2}{x}y$.

(a) Find $L[2x^2 - 3x]$ 

(b) Find $y$ such that $L[y] = \frac{e^{4x}}{x^2}$
2.3. SEPARABLE EQUATIONS

(Text, Section 2.3)

\[ y' = f(x, y) \]

is a separable equation if \( f \) can be factored into

\[ f(x, y) = p(x)h(y) \]

where \( p \) and \( h \) are continuous functions.

\[ y' = p(x)h(y) \]

is called the "standard form."
Example 1: Show that

\[ y' = xy^2 - x - y^2 + 1 \]

is separable
Solution Method:

Step 1. **Identify**: Establish that the equation IS separable.

Step 2. Divide both sides by \( h(y) \) to “separate” the variables.

\[
\frac{1}{h(y)} y' = p(x) \quad \text{or} \quad q(y)y' = p(x)
\]

which, can be written as

\[
q(y) \frac{dy}{dx} = p(x) \quad \text{and} \quad q(y)dy = p(x)dx
\]

the variables are “separated.”
Step 3. Integrate

\[ q(y) \, dy = p(x) \, dx \]

\[ \int q(y) \, dy = \int p(x) \, dx + C \]

\[ Q(y) = P(x) + C \]

where \( Q'(y) = q(y) \), \( P'(x) = p(x) \)
Note:

\[ Q(y) = P(x) + C \] is the general solution. Typically, this is an implicit relation between \( x \) and \( y \); you may or may not be able to solve it for \( y \), but you should simplify as much as possible!
Examples:

2. Find the general solution:

\[ y' = \frac{xy^2 + 4x}{2y} \]
Graphs: $y^2 = Ce^{x^2/2} - 4$
Note: If you solve for $y$

\[ y = \sqrt{Ce^{x^2/2}} - 4 \]

Graphs:
3. Find the general solution:

\[
\frac{dy}{dx} = \frac{e^x - y}{1 + e^x}
\]
4. Find the general solution:

\[
\frac{dy}{dx} = 4x\sqrt{y - 2}
\]
Singular solutions:
\[ \sqrt{y - 2} = x^2 + C \]
$y = (x^2 + C)^2 + 2$
5. Find the general solution:

\[
\frac{dy}{dx} - xy^2 = -x
\]
6. The equation

\[ y' = x(y + 2) \quad \text{or} \quad y' - xy = 2x \]

is both linear and separable. Find the general solution both ways.
Answers

1. \[ y' = (x - 1)(y^2 - 1) \]

2. \[ y^2 = Ce^{x^2/2} - 4 \]

3. \[ y = \ln \left[ \ln(1 + e^x) + C \right] \]

4. \[ \sqrt{y - 2} = x^2 + C \]

5. \[ y = \frac{1 + Ce^{x^2}}{1 - Ce^{x^2}} \]

6. \[ y = Ce^{x^2/2} - 2 \]
2.4. TRANSFORMATIONS & RELATED EQUATIONS (Text, Section 2.4)

A. Bernoulli equations

An equation that can be written in the form

\[ y' + p(x)y = q(x)y^k, \quad k \neq 0, 1 \]

is called a Bernoulli equation.

Note: (1) If \( k = 0 \) or \( 1 \), then the equation is linear.
(2) The left-hand side of a Bernoulli equation is the same as the left-hand side of a linear equation in standard form. A Bernoulli equation is "close" to being a linear equation.
Examples: \[ y' + p(x)y = q(x)y^k \]

1. Find the general solution of:

\[ y' - 4y = 2e^x \sqrt{y} \]
**Solution Method:** The change of variable

\[ v = y^{1-k}, \quad v' = (1 - k)y^{-k}y' \]

transforms the Bernoulli equation

\[ y' + p(x)y = q(x)y^k \]

into

\[ v' + (1 - k)p(x)v = (1 - k)q(x). \]

which is:

\[ v' + P(x)v = Q(x) \]

**a linear equation in** \( x \) **and** \( v \)
2. Find the general solution of:

\[ xy' + y = 2x^4 y^3 \]
3. Find the general solution of:

\[ xy' = \frac{4e^{2x}}{x^3y} - 2y \]
4. Find the general solution of:

\[ xyy' = x^2 + 2y^2 \]
Answers

1. \( y^{1/2} = Ce^{2x} - e^x \)

2. \( y^2 = \frac{1}{Cx^2 - 2x^4} \)

3. \( y = \frac{2e^{2x} + C}{x^4} \)

4. \( y^2 = Cx^4 - x^2 \)
B. Homogeneous equations

\[ y' = f(x, y) \quad (1) \]

is a **homogeneous equation** if

\[ f(tx, ty) = f(x, y) \]
**Solution Method:** The change of dependent variable

\[ y = vx, \quad y' = v + xv' \]

transforms a homogeneous equation into a separable equation:

\[ y' = f(x, y) \rightarrow v + xv' = f(x, vx) = f(1, v) \]

which can be written

\[ \frac{1}{f(1, v) - v} \, dv = \frac{1}{x} \, dx; \]

the variables are separated.
Examples:

1. Find the general solution:

\[ y' = \frac{x^2 + y^2}{2xy} \]
2. Find the general solution:

\[
\frac{dy}{dx} = \frac{x^2e^{y/x} + y^2}{xy}
\]
3. Find the general solution of:

\[ xy' = y + \sqrt{x^2 - y^2} \]
4. Find the general solution of:

\[ xyy' = x^2 + 2y^2 \]  
(same as Bernoulli Pb 4)
Answers

1. \( y^2 = x^2 - Cx \)

2. \( y + x = e^{y/x} [Cx - x \ln x] \)

3. \( y = x \sin (\ln x + C) \)

4. \( y^2 = Cx^4 - x^2 \)
Signals: homogeneous equations:

• If the equation contains a term such as $e^{y/x}$, $\sin(y/x)$, $\cos(y/x)$, etc., then the equation is likely to be homogeneous;

• If $f$ is an algebraic expression

\[ f(x, y) = \frac{P(x, y)}{Q(x, y)}, \]

and all the terms have the same degree, then the equation is homogeneous.