Second Order Linear Differential Equations

A second order linear differential equation is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where $p$, $q$, and $f$ are continuous functions on some interval $I$.

The functions $p$ and $q$ are called the coefficients of the equation.
The function $f$ is called the **forcing function** or the **nonhomogeneous term**.
“Linear”

Set \( L[y] = y'' + p(x)y' + q(x)y \).

Then, for any two twice differentiable functions \( y_1(x) \) and \( y_2(x) \),

\[
L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]
\]

and, for any constant \( c \),

\[
L[cy(x)] = cL[y(x)].
\]

That is, \( L \) is a linear differential operator.
\[ L[y] = y'' + py' + qy \]

\[ L[y_1 + y_2] = \]

\[ (y_1 + y_2)'' + p (y_1 + y_2)' + q (y_1 + y_2) \]
\[ = y_1'' + y_2'' + p (y_1' + y_2') + q (y_1 + y_2) \]
\[ = y_1'' + y_2'' + py_1' + py_2' + qy_1 + qy_2 \]
\[ = (y_1'' + py_1' + qy_1) + (y_2'' + py_2' + qy_2) \]
\[ = L[y_1] + L[y_2] \]

\[ L[cy] = (cy)'' + p(cy)' + q(cy) \]
\[ = cy'' + pcy' + qcy = c (y'' + py' + qy) \]
\[ = cL[y] \]
THEOREM: Given the second order linear equation (1). Let $a$ be any point on the interval $I$, and let $\alpha$ and $\beta$ be any two real numbers. Then the initial-value problem

$$y'' + p(x) y' + q(x) y = f(x),$$

$$y(a) = \alpha, \ y'(a) = \beta$$

has a unique solution.
The linear differential equation

\[ y'' + p(x) y' + q(x) y = f(x) \quad (1) \]

is homogeneous if the function \( f \) on the right side is \( 0 \) for all \( x \in I \). That is,

\[ y'' + p(x) y' + q(x) y = 0. \]

is a linear homogeneous equation.
If $f$ is not the zero function on $I$, that is, if $f(x) \neq 0$ for some $x \in I$, then

$$y'' + p(x)y' + q(x)y = f(x)$$

is a **linear nonhomogeneous** equation.
Section 3.2. Homogeneous Equations

\[ y'' + p(x) y' + q(x) y = 0 \quad (H) \]

where \( p \) and \( q \) are continuous functions on some interval \( I \).

The zero function, \( y(x) = 0 \) for all \( x \in I \), \( (y \equiv 0) \) is a solution of \((H)\).

The zero solution is called the trivial solution. Any other solution is a non-trivial solution.
Recall, Example 8, Chap. 1, pg 19:

Find a value of \( r \), if possible, such that \( y = x^r \) is a solution of

\[
y'' - \frac{1}{x} y' - \frac{3}{x^2} y = 0.
\]

\( y \equiv 0 \) is a solution (trivial)

\[
y_1 = x^{-1}, \quad y_2 = x^3 \text{ are solutions}
\]
Basic Theorems

**THEOREM 1:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then

\[
u(x) = y_1(x) + y_2(x)\]

is also a solution of (H).

The sum of any two solutions of (H) is also a solution of (H). (Some call this property the *superposition principle*).
Proof:

$y_1$ and $y_2$ are solutions. Therefore,

$$L[y_1] = 0 \quad \text{and} \quad L[y_2] = 0$$

$L$ is linear. Therefore
THEOREM 2: If \( y = y(x) \) is a solution of \( (H) \) and if \( C \) is any real number, then

\[
u(x) = Cy(x)\]

is also a solution of \( (H) \).

Proof: \( y \) is a solution means \( L[y] = 0 \).

\( L \) is linear:

Any constant multiple of a solution of \( (H) \) is also a solution of \( (H) \).
DEFINITION: Let \( y = y_1(x) \) and \( y = y_2(x) \) be functions defined on some interval \( I \), and let \( C_1 \) and \( C_2 \) be real numbers. The expression

\[
C_1 y_1(x) + C_2 y_2(x)
\]

is called a **linear combination** of \( y_1 \) and \( y_2 \).
Theorems 1 & 2 can be restated as:

**THEOREM 3:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), and if \( C_1 \) and \( C_2 \) are any two real numbers, then

\[
y(x) = C_1 y_1(x) + C_2 y_2(x)
\]

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).
NOTE: \[ y(x) = C_1 y_1(x) + C_2 y_2x \]

is a two-parameter family which ”looks like“ the general solution.

Is it???
Some Examples from Chapter 1:

1.

\[ y_1 = \cos 3x \text{ and } y_2 = \sin 3x \]

are solutions of

\[ y'' + 9y = 0 \quad \text{(Chap 1, p. 46)} \]

\[ y = C_1 \cos 3x + C_2 \sin 3x \]

is the general solution.
2. \( y_1 = e^{-2x} \) and \( y_2 = e^{4x} \)

are solutions of

\[ y'' - 2y' - 8y = 0 \quad \text{(Chap 1, p. 54)} \]

and

\[ y = C_1 e^{-2x} + C_2 e^{4x} \]

is the general solution.
3. \( y_1 = x \) and \( y_2 = x^3 \)

are solutions of

\[
y'' - \frac{3}{x} y' - \frac{3}{x^2} y = 0 \quad \text{(Chap 1, p. 55)}
\]

and

\[
y = C_1 x + C_2 x^3
\]

is the general solution.
Example: \[ y'' - \frac{1}{x}y' - \frac{15}{x^2}y = 0 \]

a. Solutions

\[ y_1(x) = x^5, \quad y_2(x) = 3x^5 \]

General solution:

\[ y = C_1 x^5 + C_2 (3x^5) \]

That is, is EVERY solution a linear combination of

\[ y_1 \quad \text{and} \quad y_2 \]
ANSWER: NO!!!

\[ y = x^{-3} \] is a solution AND

\[ x^{-3} \neq C_1 x^5 + C_2 (3x^5) \]

\[ C_1 x^5 + C_2 (3x^5) = M x^5 \]

\[ x^{-3} \] is NOT a constant multiple of \( x^5 \).
Now consider

\[ y_1(x) = x^5, \quad y_2(x) = x^{-3} \]

General solution: \[ y = C_1 x^5 + C_2 x^{-3} \]?

That is, is EVERY solution a linear combination of \( y_1 \) and \( y_2 \)?

Let \( y = y(x) \) be the solution of the equation that satisfies

\[ y(1) = \quad y'(1) = \]
\[ y = C_1 x^5 + C_2 x^{-3} \]
\[ y' = 5C_1 x^4 - 3C_2 x^{-4} \]

At \( x = 1 \):

\[ C_1 + C_2 = \]
\[ 5C_1 - 3C_2 = \]
In general: Let

\[ y = C_1y_1(x) + C_2y_2(x) \]

be a family of solutions of (H). When is this the general solution of (H)?

EASY ANSWER: When \( y_1 \) and \( y_2 \) ARE NOT CONSTANT MULTIPLES OF EACH OTHER.

That is, \( y_1 \) and \( y_2 \) are independent of each other.
Let

\[ y = C_1 y_1(x) + C_2 y_2(x) \]

be a two parameter family of solutions of (H). Choose any number \( a \in I \) and let \( u \) be any solution of (H).

Suppose \( u(a) = \alpha, \quad u'(a) = \beta \)
Does the system of equations

\[ C_1 y_1(a) + C_2 y_2(a) = \alpha \]

\[ C_1 y_1'(a) + C_2 y_2'(a) = \beta \]

have a unique solution??
**DEFINITION:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). The function \( W \) defined by

\[
W[y_1, y_2](x) = y_1(x)y'_2(x) - y_2(x)y'_1(x)
\]

is called the **Wronskian** of \( y_1, y_2 \).

**Determinant notation:**

\[
W(x) = 
\begin{vmatrix}
  y_1(x) & y_2(x) \\
  y'_1(x) & y'_2(x)
\end{vmatrix}
\]
THEOREM 4: Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of equation (H), and let \( W(x) \) be their Wronskian. Exactly one of the following holds:

(i) \( W(x) = 0 \) for all \( x \in I \) and \( y_1 \) is a constant multiple of \( y_2 \), AND

\[
y = C_1 y_1(x) + C_2 y_2(x)
\]

IS NOT the general solution of (H)

OR


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(ii) $W(x) \neq 0$ for all $x \in I$, AND

$$y = C_1y_1(x) + C_2y_2(x)$$

is the general solution of (H)

(Note: $W(x)$ is a solution of

$$y' + p(x)y = 0.$$  

See Section 2.1, Special Case.)

The Proof is in the text.
**Fundamental Set; Solution basis**

**DEFINITION:** A pair of solutions

\[ y = y_1(x), \quad y = y_2(x) \]

of equation (H) forms a **fundamental set of solutions** (also called a **solution basis**) if

\[ W[y_1, y_2](x) \neq 0 \quad \text{for all} \quad x \in I. \]
Section 3.3. Homogeneous Equations with Constant Coefficients

**Fact:** In contrast to first order linear equations, there are no general methods for solving

\[ y'' + p(x)y' + q(x)y = 0. \quad (H) \]

But, there is a special case of (H) for which there is a solution method, namely
\[ y'' + ay' + by = 0 \quad (1) \]

where \( a \) and \( b \) are constants.

**Solutions:** (1) has solutions of the form

\[ y = e^{rx} \]
$y = e^{rx}$ is a solution of (1) if and only if

$$r^2 + ar + b = 0 \quad (2)$$

Equation (2) is called the characteristic equation of equation (1)
Note the correspondence:

Diff. Eqn: \[ y'' + ay' + by = 0 \]

Char. Eqn: \[ r^2 + ar + b = 0 \]

The solutions \[ y = e^{rx} \] of

\[ y'' + ay' + by = 0 \]

are determined by the roots of

\[ r^2 + ar + b = 0. \]
There are three cases:

1. $r^2 + ar + b = 0$ has two, distinct real roots, $r_1 = \alpha$, $r_2 = \beta$.

2. $r^2 + ar + b = 0$ has only one real root, $r = \alpha$.

3. $r^2 + ar + b = 0$ has complex conjugate roots, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$. 
Case I: Two, distinct real roots.

\[ r^2 + ar + b = 0 \] has two distinct real roots:

\[ r_1 = \alpha, \quad r_2 = \beta, \quad \alpha \neq \beta. \]

Then

\[ y_1(x) = e^{\alpha x} \quad \text{and} \quad y_2(x) = e^{\beta x} \]

are solutions of \( y'' + ay' + by = 0. \)
\( y_1 = e^{\alpha x} \) and \( y_2 = e^{\beta x} \) are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

\[ W[y_1, y_2] = y_1 y'_2 - y_2 y'_1 = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} \]

General solution:

\[ y = C_1 e^{\alpha x} + C_2 e^{\beta x} \]
Example 1: Find the general solution of

\[ y'' - 3y' - 10y = 0. \]
Example 2: Find the general solution of

\[ y'' - 11y' + 28y = 0. \]
Case II: Exactly one real root.

\[ r = \alpha; \ (\alpha \text{ is a double root}). \] Then

\[ y_1(x) = e^{\alpha x} \]

is one solution of \( y'' + ay' + by = 0. \)

We need a second solution which is independent of \( y_1. \)
**NOTE:** In this case, the characteristic equation is

\[(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0\]

so the differential equation is

\[y'' - 2\alpha y' + \alpha^2 y = 0\]
\[ y = Ce^{\alpha x} \] is a solution for any constant \( C \). Replace \( C \) by a function \( u \) which is to be determined so that

\[ y = u(x)e^{\alpha x} \]

is a solution of:

\[ y'' - 2\alpha y' + \alpha^2 y = 0 \]

\[ y = ue^{\alpha x} \]

\[ y' = \alpha u e^{\alpha x} + e^{\alpha x} u' \]

\[ y'' = \alpha^2 ue^{\alpha x} + 2\alpha e^{\alpha x} u' + e^{\alpha x} u'' \]
$y_1 = e^{\alpha x}$ and $y_2 = xe^{\alpha x}$ are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

$$W[y_1, y_2] = y_1y_2' - y_2y_1' = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

General solution:

$$y = C_1 e^{\alpha x} + C_2 xe^{\alpha x}$$
Examples:

1. Find the general solution of

\[ y'' + 6y' + 9y = 0. \]
2. Find the general solution of

\[ y'' - 10y' + 25y = 0. \]
Case III: Complex conjugate roots.

\[ r_1 = \alpha + i \beta, \quad r_2 = \alpha - i \beta, \quad \beta \neq 0 \]

In this case

\[ u_1(x) = e^{(\alpha + i\beta)x} \quad u_2(x) = e^{(\alpha - i\beta)x} \]

are ind. solns. of \( y'' + ay' + by = 0 \) and

\[ y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \]

is the general solution. BUT, these are complex-valued functions!! **No good!!**

**We want real-valued solutions!!**
Recall from Calculus II:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^{2n}}{(2n)!} + \cdots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \pm \frac{x^{2n-1}}{(2n-1)!} + \cdots \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]

Let \( x = i\theta \), \( i^2 = -1 \)

\[ e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \]

\[ = 1 + i\theta - \frac{\theta^2}{2!} - i \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} + \cdots \]

\[ = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\theta - i \frac{\theta^3}{3!} + i \frac{\theta^5}{5!} + \cdots \]

\[ = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \right) \]
Relationships between the exponential function, sine and cosine

**Euler’s Formula:**  \( e^{i\theta} = \cos \theta + i \sin \theta \)

These follow:

\[
e^{-i\theta} = \cos \theta - i \sin \theta
\]

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}
\]

\[
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]

\[
e^{i\pi} + 1 = 0
\]
Now

\[ u_1 = e^{(\alpha + i \beta)x} = e^{\alpha x} \cdot e^{i \beta x} \]

\[ = e^{\alpha x} (\cos \beta x + i \sin \beta x) \]

\[ = e^{\alpha x} \cos \beta x + i e^{\alpha x} \sin \beta x \]

\[ u_2 = e^{(\alpha - i \beta)x} = e^{\alpha x} \cdot e^{-i \beta x} \]

\[ = e^{\alpha x} (\cos \beta x - i \sin \beta x) \]

\[ = e^{\alpha x} \cos \beta x - i e^{\alpha x} \sin \beta x \]
\{u_1 = e^{(\alpha+i\beta)x}, \; u_2 = e^{(\alpha-i\beta)x}\}

transforms into

\{y_1 = e^{\alpha x} \cos \beta x, \; y_2 = e^{\alpha x} \sin \beta x\}

\(y_1\) and \(y_2\) are not constant multiples of each other, \(\{y_1, y_2\}\) is a fundamental set,

\[
W[y_1, y_2] = y_1y'_2 - y_2y'_1 = \begin{vmatrix}
y_1(x) & y_2(x) \\
y'_1(x) & y'_2(x)
\end{vmatrix}
\]

\[= \beta e^{2\alpha x} \neq 0\]
AND

\[ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \]

is the general solution.
**Examples:** Find the general solution of

1. \( y'' - 4y' + 13y = 0. \)

2. \( y'' + 6y' + 25y = 0. \)
Comprehensive Examples:

1. Find the general solution of

\[ y'' + 6y' + 8y = 0. \]
2. Find a solution basis for

\[ y'' - 10y' + 25y = 0. \]
3. Find the solution of the initial-value problem

\[ y'' - 4y' + 8y = 0, \quad y(0) = 1, \quad y'(0) = -2. \]
5. Find the differential equation that has

\[ y = C_1 e^{2x} + C_2 e^{3x} \]

as its general solution. (See Chap 1, pg 39)
4. Find the differential equation that has

\[ y = C_1 e^{-x} + C_2 e^{4x} \]

as its general solution. (C.f. Chap 1.)
6. \( y = 5xe^{-4x} \) is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?
7. \( y = 2e^{2x} \sin 4x \) is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?
18. \[ y = C_1 e^x + C_2 e^{-2x} \].

From Exercises 1.3

19. \[ y = C_1 e^{2x} + C_2 xe^{2x} \]
22. \[ y = C_1 \cos 3x + C_2 \sin 3x. \]

24. \[ y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x. \]