Section 3.7. Higher Order Linear Differential Equations

I. BASIC TERMS (See Section 3.1)

An nth order linear differential equation is an equation of the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (L)$$

where \( p_0, p_1, \cdots, p_{n-1} \) and \( f \) are continuous functions on some interval \( I \).
(L) is **homogeneous** if \( f(x) \equiv 0 \) on \( I \):

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad \text{(H)}
\]

If \( f \) is not identically 0 in \( I \), then (L) is **nonhomogeneous**

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad \text{(N)}
\]
\[ L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \]

is a **linear (differential) operator**:

\[ L[y_1 + y_2] = L[y_1] + L[y_2] \]

\[ L[Cy] = CL[y], \quad c \quad \text{a constant} \]

Equations (H) and (N) can be written

\[ L[y] = 0 \quad \text{(H)} \]

\[ L[y] = f(x) \quad \text{(N)} \]
Existence and Uniqueness Theorem:

Let \( a \) be any point on \( I \). Let

\[ \alpha_0, \alpha_1, \cdots, \alpha_{n-1} \]

be any \( n \) real numbers. The initial-value problem:

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)
\]

\[ y(a) = \alpha_0, \ y'(a) = \alpha_1, \cdots, \ y^{(n-1)}(a) = \alpha_{n-1} \]

has a unique solution.
II. Homogeneous Equations (See Section 3.2)

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad (H) \]

The zero function, \( y(x) = 0 \) for all \( x \in I \), (\( y \equiv 0 \)) is a solution of (H). The zero solution is called the \textit{trivial solution}. Any other solution is a \textit{nontrivial} solution.
The Theorems

**THEOREM 1:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then

\[
u(x) = y_1(x) + y_2(x)
\]

is also a solution of (H).

*The sum of any two solutions of (H) is also a solution of (H).*

(Some call this property the *superposition principle*).
**THEOREM 2:** If \( y = y(x) \) is a solution of (H) and if \( C \) is any real number, then

\[
u(x) = Cy(x)
\]

is also a solution of (H).

*Any constant multiple of a solution of (H) is also a solution of (H).*
THEOREM 3: If

\[ y_1, \ y_2, \cdots, \ y_k \]

are solutions of (H) and if

\[ C_1, \ C_2, \cdots, \ C_k \]

are real numbers, then

\[ u = C_1 y_1 + C_2 y_2 + \cdots + C_k y_k \]

is a solution of (H).

Any linear combination of solutions of (H) is a solution of (H).
General Solution of (H)

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be $n$ solutions of (H). Then, for any choice of constants $C_1, C_2, \ldots, C_n$,

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) \quad \text{(GS)}$$

is a solution of (H).

Under what conditions is (GS) the general solution of (H)?
The Wronskian

Set

\[ W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-2)} & \cdots & y_n^{(n-1)} \end{vmatrix} \]

is called the **Wronskian** of \( y_1, y_2, \cdots, y_n \).
**THEOREM 4:** Let \( y_1(x), y_2(x), \ldots, y_n(x) \) be \( n \) solutions of (H) and let \( W(x) \) be their Wronskian. Exactly one of the following holds

1. \( W(x) \equiv 0 \) on \( I \) and \( y_1, y_2, \ldots, y_n \) are linearly dependent.

2. \( W(x) \neq 0 \) for all \( x \in I \) and \( y_1, y_2, \ldots, y_n \) are linearly independent. In this case

   \[
y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \quad \text{ (GS)}
   \]

   is the general solution of (H).
A set of \( n \) linearly independent solutions of (H) is called a fundamental set or a solution basis for (H).

A set of \( n \) solutions \( \{y_1, y_2, \ldots, y_n\} \) is a fundamental set if and only if their Wronskian \( W(x) \neq 0 \) for all \( x \in I \).
III. Homogeneous Equations with Constant Coefficients

(See Section 3.3)

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0(x)y = 0 \]  \hspace{1cm} (H)

\( y = e^{rx} \) is a solution if and only if \( r \) is a root of the polynomial equation

\[ P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0. \]

\( P(r) \) is called the characteristic polynomial.

\( P(r) = 0 \) is called the characteristic equation.
Linear Independence of Solutions

1. If \( r_1, r_2, \cdots, r_k \) are distinct numbers, then
   \[
y_1 = e^{r_1 x}, \; y_2 = e^{r_2 x}, \; \cdots, \; y_k = e^{r_k x}
   \]
   are linearly independent functions.

2. For any number \( a \), the functions
   \[
y_1 = e^{ax}, \; y_2 = xe^{ax}, \; \cdots, \; y_m = x^{m-1}e^{mx}
   \]
   are linearly independent functions.

3. If \( \alpha + i\beta, \alpha - i\beta \) are complex conjugates, then
   \[
y_1 = e^{\alpha x} \cos \beta x, \; y_2 = e^{\alpha x} \sin \beta x, \; y_3 = xe^{\alpha x} \cos bx, \; \cdots
   \]
   are linearly independent functions.
Examples:

1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]
2. Find the general solution of:

\[ y''' + 5y'' + 7y' + 3y = 0 \]
3. Find a fundamental set of solutions of:

\[ y^{(4)} - 5y'' - 36y = 0 \]
1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]

Hint: \( r = 2 \) is a root of the characteristic equation.
2. Find the general solution of:

\[ y''' + 5y'' + 7y' + 3y = 0 \]

Hint: \( r = -3 \) is a root of the char. poly.
4. Find the general solution of:

\[ y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0 \]

Hint: \( r = -1 + 3i \) is a root of the char. poly.
5. \[ y = C_1 e^{2x} + C_2 e^{-4x} + C_3 \cos 3x + C_4 \sin 3x \]
is the general solution of a homogeneous equation with constant coefficients. What is the equation?
6. \( y = 2e^{-x} - 3\sin 4x + 2x + 5 \) is a solution of a homogeneous equation with constant coefficients. What is the equation of least order having this solution?
Answers:  1. \[ y = C_1 e^{-4x} + C_2 e^{-x} + C_3 e^{2x} \]

2. \[ y = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{-3x} \]

3. \[ \{ e^{-3x}, e^{3x}, \cos 2x, \sin 2x \} \]

4. \[ y = C_1 e^{-x} + C_2 e^x + C_3 e^{-x} \cos 3x + C_4 e^{-x} \sin 3x \]

5. \[ y^{(4)} + 2y''' + y'' + 18y' - 72y = 0 \]

6. \[ y^{(5)} + y^{(4)} + 16y''' + 16y'' = 0 \]
IV. Nonhomogeneous Equations (See Sections 3.4, 3.5)

Given the nonhomogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]  \hspace{1cm} (N)

The corresponding homogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]  \hspace{1cm} (H)

is called the reduced equation of (N).
THEOREM 5: If \( z_1(x) \) and \( z_2(x) \) are solutions of \( (N) \), then

\[
y = z_1(x) - z_2(x)
\]

is a solution of the reduced equation \( (H) \).
THEOREM 6: Let $y_1(x), y_2(x), \cdots, y_n(x)$ be a fundamental set of solutions of (H) and $z(x)$ be a particular solution of (N). Then

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x)$$

is the general solution of (N).
V. Finding a particular solution $z$ of $(N)$:

1. **Variation of Parameters** (In theory, we can do this. In practice, difficult.)

2. **Undetermined Coefficients** (This is what we will use.)
A particular solution of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = f(x)$

<table>
<thead>
<tr>
<th>If $f(x) =$</th>
<th>try $z(x) =$*</th>
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<tbody>
<tr>
<td>$p(x)e^{rx}$</td>
<td>$z = P(x)e^{rx}$</td>
</tr>
<tr>
<td>$p(x)\cos \beta x + q(x)\sin \beta x$</td>
<td>$z = P(x)\cos \beta x + Q(x)\sin \beta x$</td>
</tr>
<tr>
<td>$p(x)e^{\alpha x} \cos \beta x + q(x)e^{\alpha x} \sin \beta x$</td>
<td>$z = P(x)e^{\alpha x} \cos \beta x + Q(x)e^{\alpha x} \sin \beta x$</td>
</tr>
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</table>

*Note: If $z$ satisfies the reduced equation, try $xz$; if $xz$ also satisfies the reduced equation, then try $x^2z$ ...
7. Find the general solution of

\[ y''' + 4y'' - 3y' - 18y = 10e^{2x} + 9x \]

Hint: 2 is a root of the characteristic polynomial
8. Give the form of a particular solution of

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 2x + 6 \]

Hint: \(-2\) is a root of the char. poly.
9. Give the form of a particular solution of

\[ y^{(4)} + 2y'' + y = 4 \cos x - 2e^{-x} + 5x - 3 \]
10. Give the form of the general solution of

\[ y^{(4)} - 16y = 2 \cos 2x - (3x + 5)e^{2x} + 3x + 1 \]
Give the form of the general solution of

\[ y''' - y'' - y' + y = 2xe^{-x} + e^x + 5x \]

Hint: 1 is a root of the characteristic polynomial
12. Give the form of the general solution of

\[ y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3 \sin 2x \]

Hint: 2 is a root of the characteristic polynomial
13. Give the form of the general solution of

\[ y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 2xe^{3x} + 5x \]

Hint: 3 is a root of the characteristic polynomial
14. Give the form of a particular solution of

\[ y''' - 3y'' + 3y' - y = (2x + 1)e^x + 10 \]
Answers

7. \[ y = C_1e^{2x} + C_2e^{-3x} + C_3xe^{-3x} + \frac{2}{5}xe^{2x} - \frac{1}{2}x + \frac{1}{12} \]

8. \[ z = Ax^2e^{-2x} + Bx \cos 3x + Cx \sin 3x + D \]

9. \[ z = Ax^2 \cos x + Bx^2 \sin x + Ce^{-x} + Dx + E \]

10. \[ y = C_1e^{2x} + C_2e^{-2x} + C_3 \cos 2x + C_4 \sin 2x + Ax \cos 2x + Bx \sin 2x + (Cx^2 + Dx)e^{2x} + Ex + F \]
11. \[ y = C_1 e^x + C_2 xe^x + C_3 e^{-x} + (Ax^2 + Bx)e^{-x} + Cx^2e^x + Dx + E \]

12. \[ y = C_1 e^{2x} + C_2 xe^{2x} + C_2 e^{-3x} + Axe^{-3x} + Be^{-2x} + (Cx^3 + Dx^2)e^{2x} + E \cos 5x + F \sin 5x \]

13. \[ y = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x + C_5 e^{3x} + (Ax^2 + Bx)e^{3x} + (Cx^3 + Bx^2) \]

14. \[ z = (Ax^4 + Bx^3)e^x + C \]