Section 3.7. Higher Order Linear Differential Equations

I. BASIC TERMS

An nth order linear differential equation is an equation of the form:

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (L) \]

where \( p_0, p_1, \cdots, p_{n-1} \) and \( f \) are continuous functions on some interval \( I \).
(L) is **homogeneous** if \( f(x) \equiv 0 \) on \( I \):

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0
\]  \( \text{(H)} \)

If \( f \) is not identically 0 in \( I \), then (L) is **nonhomogeneous**

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)
\]  \( \text{(N)} \)
\[ L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \]

is a **linear (differential) operator**:

\[ L[y_1 + y_2] = L[y_1] + L[y_2] \]

\[ L[Cy] = CL[y], \quad c \text{ a constant} \]

Equations (H) and (N) can be written

\[ L[y] = 0 \quad \text{(H)} \]

\[ L[y] = f(x) \quad \text{(N)} \]
Existence and Uniqueness Theorem:

Let $a$ be any point on $I$. Let

$$
\alpha_0, \alpha_1, \cdots, \alpha_{n-1}
$$

be any $n$ real numbers. The initial-value problem:

$$
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad \text{(N)}
$$

$$
y(a) = \alpha_0, \ y'(a) = \alpha_1, \cdots, \ y^{(n-1)}(a) = \alpha_{n-1}
$$

has a **unique** solution.
II. Homogeneous Equations (See Section 3.2)

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad \text{(H)} \]

The zero function, \( y(x) = 0 \) for all \( x \in I \), \(( y \equiv 0)\) is a solution of (H). The zero solution is called the \textbf{trivial solution}. Any other solution is a \textbf{nontrivial} solution.
The Theorems

**THEOREM 1**: If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then

\[
u(x) = y_1(x) + y_2(x)\]

is also a solution of (H).

The sum of any two solutions of (H) is also a solution of (H). (Some call this property the *superposition principle*).
**THEOREM 2:** If $y = y(x)$ is a solution of (H) and if $C$ is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

Any constant multiple of a solution of (H) is also a solution of (H).
THEOREM 3: If

\[ y_1, y_2, \ldots, y_k \]

are solutions of (H) and if

\[ C_1, C_2, \ldots, C_k \]

are real numbers, then

\[ u = C_1y_1 + C_2y_2 + \cdots + C_ky_k \]

is a solution of (H).

Any linear combination of solutions of (H) is a solution of (H).
General Solution of (H)

Let $y_1(x), y_2(x), \cdots, y_n(x)$ be $n$ solutions of (H). Then, for any choice of constants $C_1, C_2, \cdots, C_n$,

$$
y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) \quad \text{(GS)}$$

is a solution of (H).

Under what conditions is (GS) the general solution of (H)?
The Wronskian

Set

\[ W(x) = \begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{(n-1)} & y_{(n-2)} & \cdots & y_{(n-1)} \\
\end{vmatrix} \]

is called the **Wronskian** of \( y_1, y_2, \cdots, y_n \).
**Theorem 4**: Let \( y_1(x), y_2(x), \ldots, y_n(x) \) be \( n \) solutions of \((H)\) and let \( W(x) \) be their Wronskian. Exactly one of the following holds

1. \( W(x) \equiv 0 \) on \( I \) and \( y_1, y_2, \ldots, y_n \) are linearly dependent.

2. \( W(x) \not\equiv 0 \) for all \( x \in I \) and \( y_1, y_2, \ldots, y_n \) are linearly independent. In this case

\[
y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) \quad \text{(GS)}
\]

is the general solution of \((H)\).
A set of $n$ linearly independent solutions of (H) is called a fundamental set or a solution basis for (H).

A set of $n$ solutions $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set if and only if their Wronskian $W(x) \neq 0$ for all $x \in I$. 
III. Homogeneous Equations with Constant Coefficients

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0(x)y = 0 \quad \text{(H)} \]

\[ y = e^{rx} \] is a solution if and only if \( r \) is a root of the polynomial equation

\[ P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0. \]

\( P(r) \) is called the \textbf{characteristic polynomial}.

\( P(r) = 0 \) is called the \textbf{characteristic equation}. 
Linear Independence of Solutions

1. If \( r_1, r_2, \cdots, r_k \) are distinct numbers, then

\[
y_1 = e^{r_1x}, \ y_2 = e^{r_2x}, \cdots, \ y_k = e^{r_kx}
\]

are linearly independent functions.

2. For any number \( a \), the functions

\[
y_1 = e^{ax}, \ y_2 = xe^{ax}, \cdots, \ y_m = x^{m-1}e^{mx}
\]

are linearly independent functions.

3. If \( \alpha + i\beta, \alpha - i\beta \) are complex conjugates, then

\[
y_1 = e^{\alpha x} \cos \beta x, \ y_2 = e^{\alpha x} \sin \beta x, \ y_3 = xe^{ax} \cos bx, \cdots
\]

are linearly independent functions.
Examples:

1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]

Char eqn \( n^3 + 3n^2 - 6n - 8 = 0 \)

\( (n-2)(n+1)(n+4) = 0 \)

Fund set \( y_1 = e^{2x}, \quad y_2 = e^{-x}, \quad y_3 = e^{-4x} \)

\[ y = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-4x} \quad \text{Gen Soln} \]
2. Find the general solution of:

\[ y'''' + 5y''' + 7y'' + 3y = 0 \]

\[ n^3 + 5n^2 + 7n + 3 = 0 \]

\[ (n+1)(n+2)(n+3) = 0 \]

Fundamental set:

\[ y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = e^{-3x} \]

\[ y = c_1 e^x + c_2 xe^x + c_3 e^{-3x} \quad \text{Gen Sol.} \]
3. Find the general solution of:

\[ y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0 \]

\[ (\lambda - 1)(\lambda^2 + 2\lambda + 10) = 0 \]

The solutions are:

\[ e^x, e^{-x}, e^{3x}, e^{-3x} \]

\[ y = c_1e^x + c_2e^{-x} + c_3e^{3x} + c_4e^{-3x} \]
4. Find the general solution of:

\[ y''' + 5y'' + 7y' + 3y = 0 \]

Hint: \( r = -3 \) is a root of the char. poly.

\[ (r + 3)(r^2 + c r + c) = 0 \]

\( n + 3 \) is a factor

Sols: \( e^{-3x}, e^{-x}, x e^{-x} \)

Gen Soln \( y = c_1 e^{-3x} + c_2 e^{-x} + c_3 x e^{-x} \)
Find the general solution of:

\[ y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0 \]

Hint: \( r = -1 + 3i \) is a root of the char. poly.

\[
\begin{align*}
r + 2r^3 + 9r^2 - 2r - 10 &= (r-1)(r + 2r + 10) \\
(r-(-1+3i))(r-(-1-3i)) &= r^2 + 2r + 10 \\
\frac{r^2 - 1}{1} &= \\ \\
r^4 + 2r^3 + 10r^2 &= \\
\overbrace{r^4 + 2r^3 + 9r^2 - 2r - 10}^{\text{char. poly.}} \\
-2r - 10 \\
\end{align*}
\]

Roots: \( -1, -1 \pm 3i \)

\[ e^x, e^{-x}, e^{3x}, e^{-3x}, \cos 3x, \sin 3x \]

General Solution:

\[ y = C_1 e^x + C_2 e^{-x} + C_3 e^{3x} \cos 3x + C_4 e^{-3x} \sin 3x \]
Find the general solution of:

\[ y^{(4)} - 5y'' - 36y = 0 \]

\[ \lambda - 5 \lambda^2 - 36 = 0 \]

\[ \lambda - 5 \lambda - 36 = 0 \]

\[ (\lambda - 9)(\lambda + 4) = 0 \]

\[ \lambda_1 = 9, \lambda_2 = -4, 2i, -2i \]

\[ y = C_1e^{3x} + C_2e^{-3x} + C_3\cos 2x + C_4\sin 2x \]
5. \[ y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x \]
is the general solution of a homogeneous equation. What's the equation?

\[
\begin{align*}
\lambda^2 - 4 &= 0 \\
(\lambda - 2)(\lambda + 2) &= 0 \\
\Rightarrow \lambda &= 2, -2
\end{align*}
\]

2 roots: \(2, -2\)

\[ y^{(4)} - 16y = 0 \]
8. (a) \( y = 2e^{-x} - 3\sin 4x + 2x + 5 \) is a solution of a homogeneous equation. What is the equation of least order having this solution?

Least order
\[
(n+1)(n^2+16) y^{(3)} + (n+1)(n^2+16) y^{(2)} + (5)(4) y^{(1)} + 16 y^{(1)} + 16 y = 0
\]
IV. Nonhomogeneous Equations

Given the nonhomogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]  \hspace{1cm} (N)

The corresponding homogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]  \hspace{1cm} (H)

is called the reduced equation of (N).
THEOREM 5: If \( z_1(x) \) and \( z_2(x) \) are solutions of \( (N) \), then

\[ y = z_1(x) - z_2(x) \]

is a solution of the reduced equation \( (H) \).

\[
L \left[ z_1 \right] = f, \quad L \left[ z_2 \right] = f \\
L \left[ z_1 - z_2 \right] = L \left[ z_1 \right] - L \left[ z_2 \right] = f - f > 0
\]
THEOREM 6: Let \( y_1(x), y_2(x), \ldots, y_n(x) \) be a fundamental set of solutions of (H) and \( z(x) \) be a particular solution of (N). Then

\[
y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x)
\]

is the general solution of (N).
V. Finding a particular solution $z$ of (N):

1. Variation of Parameters (In theory, we can do this. In practice, difficult.)

2. Undetermined Coefficients (This is what we will use.)
A particular solution of \( y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = f(x) \)

<table>
<thead>
<tr>
<th>If ( f(x) = )</th>
<th>try ( z(x) = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x)e^{rx} )</td>
<td>( z = P(x)e^{rx} )</td>
</tr>
<tr>
<td>( p(x)\cos \beta x + q(x)\sin \beta x )</td>
<td>( z = P(x)\cos \beta x + Q(x)\sin \beta x )</td>
</tr>
<tr>
<td>( p(x)e^{\alpha x}\cos \beta x + q(x)e^{\alpha x}\sin \beta x )</td>
<td>( z = P(x)e^{\alpha x}\cos \beta x + Q(x)e^{\alpha x}\sin \beta x )</td>
</tr>
</tbody>
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*Note: If \( z \) satisfies the reduced equation, try \( xz \); if \( xz \) also satisfies the reduced equation, then try \( x^2z \) ...
7. Give the form of a particular solution of

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 2x + 6 \]

\[ z = Ae^{-2x} + B \cos 2x + C \sin 2x + D \]

Check reduced eqn

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 0 \]

Evaluate characteristic polynomial:

\[ -2 | 1 4 13 36 36 \\
    -2 | -2 -8 -18 -36 \]

\[ -2 | 1 2 9 18 \text{ (0)} \]

\[ -2 | 1 0 -9 \text{ (0)} \]

So \[ z = Axe^{-2x} + B \cos 2x + C \sin 2x + D \]
7. Give the form of a particular solution of

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 3x + 6 \]

Answer: 

\[ z = Ax^2 e^{-2x} + Bx \cos 3x + Cx \sin 3x + D \]
8. Give the form of a particular solution of

\[ y^{(4)} + 2y'' + y = 4\cos x - 2e^{-x} + x \]

\[ t = A\cos x + B\sin x + Ce^{-x} + Dx + E \]

Check reduced case

\[ y^{(4)} + 2y'' + y = 0 \]

\[ \lambda^4 + 2\lambda^2 + 1 = 0 \]

\[ (\lambda^2 + 1)^2 = 0 \]

\[ \cos x, \sin x, x\cos x, x\sin x \]

\[ z = A\lambda^2 x\cos x + B\lambda^2 x\sin x + C e^{-x} + Dx + E \]
8. Give the form of a particular solution of

\[ y^{(4)} + 2y'' + y = 4\cos x - 2e^{-x} + x \]

**Answer:**

\[ z = Ax^2 \cos x + Bx^2 \sin x + Ee^{-x} + Fx + G \]

which is the same as

\[ y = Ax^2 \cos x + Bx^2 \sin x + C e^{-x} + Dx + E \]
9. Give the form of the general solution of

\[ y^{(4)} - 16y = 2 \cos 2x - (3x + 5)e^{2x} + 3x + 1 \]

\[ z = A \cos 2x + B \sin 2x + (Cx + D)e^{2x} + Ex + F \]

Check reduced eqn

\[ y^{(4)} - 16y = 0 \]

\[ n^4 - 16 = 0 \]

\[ (n - 4)(n + 4) = 0 \]

\[ e^{2x}, e^{-2x}, \cos 2x, \sin 2x \]

\[ z = A \cos 2x + B \sin 2x + (Cx + D)e^{2x} + Ex + F \]
Example 9A

\[ y^{(5)} - 16y' = 3 \cos 2x + (4x+1)e^{2x} + 4x^2 \]

\[ t = A \cos 2x + B \sin 2x + (Cx+D)e^{2x} + (Ex+F)e^{-2x} \]

Check reduced eqn \( y^{(5)} - 16y' = 0 \)

\[ \lambda^5 - 16\lambda = \lambda(\lambda^4 - 16) \]

\[ \lambda^4 - 16 = \lambda(\lambda^3 - 16) \]

\[ \lambda^3 - 16 = \lambda(\lambda^2 - 16) \]

\[ \lambda^2 - 16 = \lambda(\lambda^2 - 16) \]

\[ \lambda - 4 = \lambda[\lambda^2 - 16] \]

\[ \lambda = \pm 4 \]

So

\[ t = A \cos 2x + B \sin 2x + (Cx+D)e^{2x} + (Ex+F)e^{-2x} \]
9. Give the form of the general solution of

\[ y^{(4)} - 16y = 2 \cos 2x - (3x + 5)e^{2x} + 3x + 1 \]

Answer:

\[ y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x + \]

\[ Ax \cos 2x + Bx \sin 2x + (C x^2 + Dx) e^{2x} + Ex + F \]
10. Give the form of the general solution of

\[ y''' - y'' - y' + y = 2xe^{-x} + e^x + 5x \]

\[ y = (Ax+B)(e^x) + Ce^x + Dx + E \]

Check reduced eqn:

\[ y''' - y'' - y' + y = 0 \]

\[ 3\lambda - \lambda^2 - \lambda + 1 = 0 \]

\[ (\lambda - 1)(\lambda - 1) = 0 \]

So, \( y = (Ax + Bx)e^x + Ce^x \)
10. Give the form of the general solution of

\[ y''' - y'' - y' + y = 2xe^{-x} + e^x + 5x \]

Answer:

\[ z = C_1e^x + C_2xe^x + C_3e^{-x} + (Ax^2 + Bx)e^{-x} + Cx^2e^x + Dx + E \]
11. Give the form of the general solution of

\[ y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3\sin 2x \]

\[ z = Ae^{-3x} + Be^{-2x} + (Cx + D)e^{2x} + E\cos 2x + F\sin 2x \]

Check reduced eqn

\[
\begin{vmatrix}
-3 & 1 & -1 & 8 & 12 \\
1 & 0 & 12 & -12 \\
0 & 0 & -12 & -12 \\
1 & 4 & 4 & 4 \\
\end{vmatrix}
\]

\[ n^3 - n^2 - 8n + 12 = 0 \]

I'm worried about

\[ (n+3)(n-2)^2 \]

\[ 2, 2, 2, \frac{2\pi}{x} \]

\[ z = Ax e^{-3x} + Be^{-2x} + (Cx + D)e^{2x} + E\cos 2x + F\sin 2x \]
11. Give the form of the general solution of

\[ y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3 \sin 5x \]

Answer:

\[ y = C_1e^{2x} + C_2xe^{2x} + C_2e^{-3x} + Axe^{-3x} + Be^{-2x} \]

\[ + (Cx^3 + Dx^2)e^{2x} + E \cos 5x + F \sin 5x \]