Section 3.7. Higher Order Linear Differential Equations

I. BASIC TERMS (See Section 3.1)

An nth order linear differential equation is an equation of the form:

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]  

(L)

where \( p_0, p_1, \cdots, p_{n-1} \) and \( f \) are continuous functions on some interval \( I \).
(L) is **homogeneous** if \( f(x) \equiv 0 \) on \( I \):

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad (H)
\]

If \( f \) is not identically 0 in \( I \), then (L) is **nonhomogeneous**

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)
\]
\[ L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \]

is a **linear (differential) operator**:

\[ L[y_1 + y_2] = L[y_1] + L[y_2] \]

\[ L[cy] = cL[y], \quad c \text{ a constant} \]

Equations (H) and (N) can be written

\[ L[y] = 0 \quad \text{(H)} \]

\[ L[y] = f(x) \quad \text{(N)} \]
Existence and Uniqueness Theorem:

Let \( a \) be any point on \( I \). Let

\[ \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \]

be any \( n \) real numbers. The initial-value problem:

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)
\]

\[
y(a) = \alpha_0, \ y'(a) = \alpha_1, \ldots, \ y^{(n-1)}(a) = \alpha_{n-1}
\]

has a **unique** solution.
II. Homogeneous Equations (See Section 3.2)

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad (H) \]

The zero function, \( y(x) = 0 \) for all \( x \in I \), \( (y \equiv 0) \) is a solution of (H). The zero solution is called the trivial solution. Any other solution is a nontrivial solution.
The Theorems

**THEOREM 1**: If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then

$$u(x) = y_1(x) + y_2(x)$$

is also a solution of (H).

*The sum of any two solutions of (H) is also a solution of (H).*

(Some call this property the *superposition principle*).
THEOREM 2: If \( y = y(x) \) is a solution of (H) and if \( C \) is any real number, then

\[
u(x) = Cy(x)
\]

is also a solution of (H).

*Any constant multiple of a solution of (H) is also a solution of (H).*
THEOREM 3: If

\[ y_1, y_2, \ldots, y_k \]

are solutions of (H) and if

\[ C_1, C_2, \ldots, C_k \]

are real numbers, then

\[ u = C_1 y_1 + C_2 y_2 + \cdots + C_k y_k \]

is a solution of (H).

Any linear combination of solutions of (H) is a solution of (H).
**General Solution of (H)**

Let \( y_1(x), y_2(x), \cdots, y_n(x) \) be \( n \) solutions of (H). Then, for any choice of constants \( C_1, C_2, \cdots, C_n, \)

\[
y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \quad \text{(GS)}
\]

is a solution of (H).

Under what conditions is (GS) the general solution of (H)?
The Wronskian

Set

\[
W(x) = \begin{vmatrix}
    y_1 & y_2 & \cdots & y_n \\
    y'_1 & y'_2 & \cdots & y'_n \\
    \vdots & \vdots & \ddots & \vdots \\
    y^{(n-1)}_1 & y^{(n-2)}_2 & \cdots & y^{(n-1)}_n
\end{vmatrix}
\]

is called the **Wronskian** of \( y_1, y_2, \cdots, y_n \).
**THEOREM 4:** Let $y_1(x), y_2(x), \cdots, y_n(x)$ be $n$ solutions of (H) and let $W(x)$ be their Wronskian. Exactly one of the following holds

1. $W(x) \equiv 0$ on $I$ and $y_1, y_2, \cdots, y_n$ are linearly dependent.

2. $W(x) \neq 0$ for all $x \in I$ and $y_1, y_2, \cdots, y_n$ are linearly independent. In this case

   $$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) \quad \text{(GS)}$$

   is the general solution of (H).
A set of \( n \) linearly independent solutions of (H) is called a **fundamental set** or a **solution basis** for (H).

A set of \( n \) solutions \( \{y_1, y_2, \ldots, y_n\} \) is a fundamental set if and only if their Wronskian \( W(x) \neq 0 \) for all \( x \in I \).
III. Homogeneous Equations with Constant Coefficients

(See Section 3.3)

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0(x)y = 0 \quad (H) \]

\[ y = e^{rx} \quad \text{is a solution if and only if} \quad r \quad \text{is a root of the polynomial equation} \]

\[ P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0. \]

\[ P(r) \quad \text{is called the characteristic polynomial}. \]

\[ P(r) = 0 \quad \text{is called the characteristic equation}. \]
Linear Independence of Solutions

1. If \( r_1, r_2, \ldots, r_k \) are distinct numbers, then

\[
y_1 = e^{r_1 x}, \ y_2 = e^{r_2 x}, \cdots, \ y_k = e^{r_k x}
\]

are linearly independent functions.

2. For any number \( a \), the functions

\[
y_1 = e^{ax}, \ y_2 = xe^{ax}, \cdots, \ y_m = x^{m-1}e^{mx}
\]

are linearly independent functions.

3. If \( \alpha + i\beta, \ \alpha - i\beta \) are complex conjugates, then

\[
y_1 = e^{\alpha x} \cos \beta x, \ y_2 = e^{\alpha x} \sin \beta x, \ y_3 = xe^{ax} \cos bx, \cdots
\]

are linearly independent functions.
Examples:

1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]
2. Find the general solution of:

\[ y'''' + 5y'' + 7y' + 3y = 0 \]
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\[ y''' + 3y'' - 6y' - 8y = 0 \]

Hint: \( r = 2 \) is a root of the characteristic equation.
2. Find the general solution of:

\[ y''' + 5y'' + 7y' + 3y = 0 \]

Hint: \( r = -3 \) is a root of the char. poly.
3. Find the general solution of:

\[ y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0 \]

Hint: \( r = -1 + 3i \) is a root of the char. poly.
4. Find the general solution of:

\[ y^{(4)} - 5y'' - 36y = 0 \]
5.

\[ y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x \]

is the general solution of a homogeneous equation. What's the equation?
6. \[ y = 2e^{-x} - 3\sin 4x + 2x + 5 \]

is a solution of a homogeneous equation. What is the equation of least order having this solution?
IV. Nonhomogeneous Equations (See Sections 3.4, 3.5)

Given the nonhomogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \]  \hfill (N)

The corresponding homogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]  \hfill (H)

is called the reduced equation of (N).
THEOREM 5: If \( z_1(x) \) and \( z_2(x) \) are solutions of (N), then

\[
y = z_1(x) - z_2(x)
\]

is a solution of the reduced equation (H).
**THEOREM 6**: Let \( y_1(x), y_2(x), \ldots, y_n(x) \) be a fundamental set of solutions of \((H)\) and \( z(x) \) be a particular solution of \((N)\). Then

\[
y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x)
\]
is the general solution of \((N)\).
V. Finding a particular solution $z$ of $(N)$:

1. Variation of Parameters (In theory, we can do this. In practice, difficult.)

2. Undetermined Coefficients (This is what we will use.)
A particular solution of \( y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = f(x) \)

<table>
<thead>
<tr>
<th>If ( f(x) = )</th>
<th>try ( z(x) = )</th>
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</thead>
<tbody>
<tr>
<td>( p(x)e^{rx} )</td>
<td>( z = P(x)e^{rx} )</td>
</tr>
<tr>
<td>( p(x) \cos \beta x + q(x) \sin \beta x )</td>
<td>( z = P(x) \cos \beta x + Q(x) \sin \beta x )</td>
</tr>
<tr>
<td>( p(x)e^{\alpha x} \cos \beta x + q(x)e^{\alpha x} \sin \beta x )</td>
<td>( z = P(x)e^{\alpha x} \cos \beta x + Q(x)e^{\alpha x} \sin \beta x )</td>
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*Note: If \( z \) satisfies the reduced equation, try \( xz \); if \( xz \) also satisfies the reduced equation, then try \( x^2z \) …
7. Find the general solution of

\[ y''' + 4y'' - 3y' - 18y = 10e^{2x} + 9x \]
8. Give the form of a particular solution of

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 2x + 6 \]
9. Give the form of a particular solution of

\[ y^{(4)} + 2y'' + y = 4\cos x - 2e^{-x} + 5x - 3 \]
10. Give the form of the general solution of

\[ y^{(4)} - 16y = 2 \cos 2x - (3x + 5)e^{2x} + 3x + 1 \]
11. Give the form of the general solution of

\[ y''' - y'' - y' + y = 2xe^{-x} + e^x + 5x \]
12. Give the form of the general solution of

\[ y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3 \sin 2x \]
13. Give the form of the general solution of

\[ y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 2xe^{3x} + 5x \]
14. Give the form of a particular solution of

\[ y''' - 3y'' + 3y' - y = (2x + 1)e^x + 10 \]
Answers

7. \[ y = C_1 e^{2x} + C_2 e^{-3x} + C_3 xe^{-3x} + \frac{2}{5}xe^{2x} - \frac{1}{2}x + \frac{1}{12} \]

8. \[ z = Ax^2 e^{-2x} + Bx \cos 3x + Cx \sin 3x + D \]

9. \[ z = Ax^2 \cos x + Bx^2 \sin x + Ce^{-x} + Dx + E \]

10. \[ y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x + Ax \cos 2x + Bx \sin 2x + (Cx^2 + Dx)e^{2x} + Ex + F \]
11. \[ y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + (A x^2 + B x) e^{-x} + C x^2 e^x + D x + E \]

12. \[ y = C_1 e^{2x} + C_2 x e^{2x} + C_2 e^{-3x} + A x e^{-3x} + B e^{-2x} + (C x^3 + D x^2) e^{2x} + E \cos 5x + F \sin 5x \]

13. \[ y = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x + C_5 e^{3x} + (A x^2 + B x) e^{3x} + (C x^3 + B x^2) \]

14. \[ z = (A x^4 + B x^3) e^x + C \]