Section 3.7. Higher Order Linear Differential Equations

I. BASIC TERMS (See Section 3.1)

An nth order linear differential equation is an equation of the form:

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (L) \]

where \( p_0, p_1, \cdots, p_{n-1} \) and \( f \) are continuous functions on some interval \( I \).
(L) is **homogeneous** if \( f(x) \equiv 0 \) on \( I \):

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad (H)
\]

If \( f \) is not identically 0 in \( I \), then (L) is **nonhomogeneous**

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)
\]
\[ L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y \]

is a **linear (differential) operator**:

\[
L[y_1 + y_2] = L[y_1] + L[y_2]
\]

\[
L[Cy] = CL[y], \quad c \text{ a constant}
\]

Equations (H) and (N) can be written

\[ L[y] = 0 \] (H)

\[ L[y] = f(x) \] (N)
Existence and Uniqueness Theorem:

Let \( a \) be any point on \( I \). Let

\[ \alpha_0, \alpha_1, \cdots, \alpha_{n-1} \]

be any \( n \) real numbers. The initial-value problem:

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad (N)
\]

\[ y(a) = \alpha_0, \ y'(a) = \alpha_1, \cdots, \ y^{(n-1)}(a) = \alpha_{n-1} \]

has a unique solution.
II. HOMOGENEOUS EQUATIONS (See Section 3.2)

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad \text{(H)} \]

The zero function, \( y(x) = 0 \) for all \( x \in I \), \( (y \equiv 0) \) is a solution of (H). The zero solution is called the **trivial solution**. Any other solution is a **nontrivial** solution.
The Theorems:

**THEOREM 1:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then

\[
u(x) = y_1(x) + y_2(x)
\]

is also a solution of (H).

*The sum of any two solutions of (H) is also a solution of (H).*

(Some call this property the *superposition principle*).
**THEOREM 2:** If $y = y(x)$ is a solution of (H) and if $C$ is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

*Any constant multiple of a solution of (H) is also a solution of (H).*
**THEOREM 3:** If

\[ y_1, \, y_2, \, \cdots, \, y_k \]

are solutions of \((H)\) and if

\[ C_1, \, C_2, \, \cdots, \, C_k \]

are real numbers, then

\[ u = C_1 y_1 + C_2 y_2 + \cdots + C_k y_k \]

is a solution of \((H)\).

*Any linear combination of solutions of \((H)\) is a solution of \((H)\).*
General Solution of (H)

Let $y_1(x), y_2(x), \cdots, y_n(x)$ be $n$ solutions of (H). Then, for any choice of constants $C_1, C_2, \cdots, C_n$,

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) \quad \text{(GS)}$$

is a solution of (H).

Under what conditions is (GS) the general solution of (H)?
The Wronskian

Set

\[ W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_{(n-1)} & y_{(n-2)} & \cdots & y_{(n-1)} \end{vmatrix} \]

is called the Wronskian of \( y_1, y_2, \cdots, y_n \).
**THEOREM 4:** Let $y_1(x), y_2(x), \cdots, y_n(x)$ be $n$ solutions of (H) and let $W(x)$ be their Wronskian. Exactly one of the following holds

1. $W(x) \equiv 0$ on $I$ and $y_1, y_2, \cdots, y_n$ are linearly dependent.

2. $W(x) \neq 0$ for all $x \in I$ and $y_1, y_2, \cdots, y_n$ are linearly independent. In this case

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \quad \text{(GS)}$$

is the general solution of (H).
A set of \( n \) linearly independent solutions of (H) is called a fundamental set or a solution basis for (H).

A set of \( n \) solutions \( \{y_1, y_2, \cdots, y_n\} \) is a fundamental set if and only if their Wronskian \( W(x) \neq 0 \) for all \( x \in I \).
III. HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS (See Section 3.3)

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \quad \text{(H)} \]

\( y = e^{rx} \) is a solution if and only if \( r \) is a root of the polynomial equation

\[ P(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0. \]

\( P(r) \) is called the characteristic polynomial.

\( P(r) = 0 \) is called the characteristic equation.
Example:  \( y^{(4)} - 5y'' - 36y = 0 \)

Char. eqn.:  \( r^4 - 5r^2 - 36 = 0 \)

\[
r^4 - 5r^2 - 36 = (r^2 - 9)(r^2 + 4)
= (r - 3)(r + 3)(r^2 + 4) = 0
\]

Roots:  \( r = -3, \ r = 3, \ r = \pm 2i \)

Solutions:  \( y_1 = e^{-3x}, \ y_2 = e^{3x}, \ y_3 = \cos 2x, \ y_4 = \sin 2x \)
Linear Independence of Solutions

1. If $r_1, r_2, \cdots, r_k$ are distinct numbers, then
   \[ y_1 = e^{r_1x}, \; y_2 = e^{r_2x}, \; \cdots, \; y_k = e^{r_kx} \]
   are linearly independent functions.

2. For any number $a$, the functions
   \[ y_1 = e^{ax}, \; y_2 = xe^{ax}, \; \cdots, \; y_m = x^{m-1}e^{ax} \]
   are linearly independent functions.

3. If $\alpha + i\beta, \alpha - i\beta$ are complex conjugates, then
   \[ y_1 = e^{\alpha x} \cos \beta x, \; y_2 = e^{\alpha x} \sin \beta x, \; y_3 = xe^{\alpha x} \cos bx, \cdots \]
   are linearly independent functions.
Examples:

1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]
2. Find the general solution of:

\[ y^{(4)} - y''' - 7y'' + y' + 6y = 0 \]
1. Find the general solution of:

\[ y''' + 3y'' - 6y' - 8y = 0 \]

Hint: \( r = 2 \) is a root of the characteristic equation.
2. Find the general solution of:

\[ y^{(4)} - y''' - 7y'' + y' + 6y = 0 \]

Hint: \( r = 1, \ r = -1 \) are roots of the char. poly.
3. Find a fundamental set of solutions of:

\[ y^{(4)} + 5y'' - 36y = 0 \]
4. Find the general solution of:

\[ y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0 \]

Hint: \( r = -1 + 3i \) is a root of the char. poly.
5. \[ y = C_1 e^{2x} + C_2 e^{-4x} + C_3 \cos 3x + C_4 \sin 3x \]
is the general solution of a homogeneous equation with constant coefficients. What is the equation?
6. \( y = 2e^{-x} - 3 \sin 4x + 2x + 5 \) is a solution of a homogeneous equation with constant coefficients. What is the equation of least order having this solution?
Answers:

1. \( y = C_1 e^{-4x} + C_2 e^{-x} + C_3 e^{2x} \)

2. \( y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x} + C_4 e^{-2x} \)

3. \( \{ e^{-2x}, e^{2x}, \cos 3x, \sin 3x \} \)

4. \( y = C_1 e^{-x} + C_2 e^x + C_3 e^{-x} \cos 3x + C_4 e^{-x} \sin 3x \)

5. \( y^{(4)} + 2y''' + y'' + 18y' - 72y = 0 \)

6. \( y^{(5)} + y^{(4)} + 16y''' + 16y'' = 0 \)
Given the nonhomogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \quad \text{(N)} \]

The corresponding homogeneous equation

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad \text{(H)} \]

is called the **reduced equation** of (N).
THEOREM 1: If $z_1(x)$ and $z_2(x)$ are solutions of (N), then

$$ y = z_1(x) - z_2(x) $$

is a solution of the reduced equation (H).
THEOREM 2: Let $y_1(x)$, $y_2(x)$, $\ldots$, $y_n(x)$ be a fundamental set of solutions of (H) and $z(x)$ be a particular solution of (N). Then

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + z(x)$$

is the general solution of (N).
V. Finding a particular solution $z$ of $(N)$:

1. **Variation of Parameters** (In theory, we can do this. In practice, difficult.)

2. **Undetermined Coefficients** (This is what we will use.)
A particular solution of \( y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = f(x) \)

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<thead>
<tr>
<th>If</th>
<th>( f(x) = )</th>
<th>try</th>
<th>( z(x) = )*</th>
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<td>( p(x)e^{rx} )</td>
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<td>( z = P(x)e^{rx} )</td>
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<td>( p(x) \cos \beta x + q(x) \sin \beta x )</td>
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<td>( z = P(x) \cos \beta x + Q(x) \sin \beta x )</td>
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*Note: If \( z \) satisfies the reduced equation, try \( xz \); if \( xz \) also satisfies the reduced equation, then try \( x^2z \) ...
7. Find the general solution of

\[ y''' + 4y'' - 3y' - 18y = 10e^{2x} + 9x \]

Hint: 2 is a root of the characteristic polynomial
8. Give the form of a particular solution of

\[ y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{-2x} + \sin 2x + 6 \]

Hint: \(-2\) is a root of the char. poly.
9. Give the form of a particular solution of

\[ y^{(4)} + 2y'' + y = 4\cos x - 2e^{-x} + 5x - 3 \]
10. Give the form of the general solution of

\[ y^{(4)} - 16y = 2 \cos 2x - (3x + 5)e^{2x} + 3x + 1 \]
11. Give the form of the general solution of

\[ y''' - y'' - y' + y = 2xe^{-x} + e^x + 5x \]

Hint: 1 is a root of the characteristic polynomial
12. Give the form of the general solution of

\[ y''' - y'' - 8y' + 12y = -5e^{-3x} + 4e^{-2x} + xe^{2x} + 3 \sin 2x \]

Hint: 2 is a root of the characteristic polynomial
13. Give the form of the general solution of

\[ y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 2xe^{3x} + 5x \]

Hint: 3 is a root of the characteristic polynomial
14. Give the form of a particular solution of

\[ y''' - 3y'' + 3y' - y = (2x + 1)e^x + 10 \]
Answers

7. \[ y = C_1 e^{2x} + C_2 e^{-3x} + C_3 x e^{-3x} + \frac{2}{5} x e^{2x} - \frac{1}{2} x + \frac{1}{12} \]

8. \[ z = A x^2 e^{-2x} + B x \cos 3x + C x \sin 3x + D \]

9. \[ z = A x^2 \cos x + B x^2 \sin x + C e^{-x} + D x + E \]

10. \[ y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x + A x \cos 2x + B x \sin 2x + (C x^2 + D x) e^{2x} + E x + F \]
11.  \[ y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + (A x^2 + B x) e^{-x} + C x^2 e^x + D x + E \]

12.  \[ y = C_1 e^{2x} + C_2 x e^{2x} + C_3 e^{-3x} + A x e^{-3x} + B e^{-2x} + (C x^3 + D x^2) e^{2x} + E \cos 2x + F \sin 2x \]

13.  \[ y = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x + C_5 e^{3x} + (A x^2 + B x) e^{3x} + (C x^3 + D x^2) \]

14.  \[ z = (A x^4 + B x^3) e^x + C \]