Chapter 6: Systems of Linear Differential Equations

Let $a_{11}(t), a_{12}(t), \ldots, a_{nn}(t),$

$b_1(t), b_2(t), \ldots, b_n(t)$

be continuous functions on the interval $I$.

The system of $n$ first-order differential equations
\[ \begin{align*}
x'_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\
x'_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\
&\quad \vdots \\
x'_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)
\end{align*} \]

is called a **first-order linear differential system**.

The system is **homogeneous** if

\[ b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on} \quad I. \]

It is **nonhomogeneous** if the functions \( b_i(t) \) are not all identically zero on \( I \).
Set
\[ A(t) = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
  a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
   \vdots & \vdots & \ddots & \vdots \\
  a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{pmatrix} \]

and
\[ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}. \]

The system can be written in the vector-matrix form

\[ x' = A(t) x + b(t). \quad \text{(S)} \]
The matrix $A(t)$ is called the matrix of coefficients or the coefficient matrix.

The vector $b(t)$ is called the non-homogeneous term, or forcing function.
A **solution** of the linear differential system (S) is a differentiable vector function

\[ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \]

that satisfies (S) on the interval \( I \).
Example 1:

\[ x'_1 = x_1 + 2x_2 - 5e^{2t} \]
\[ x'_2 = 3x_1 + 2x_2 + 3e^{2t} \]

Vector/matrix form

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}
\]

or

\[
x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}x + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}
\]
\[ x(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} \] is a solution of

\[ x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix} \]
In fact,

\[ x(t) = C_1 \begin{pmatrix} 2e^{4t} \\ 3e^{4t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} \]

is a solution for any numbers \( C_1, C_2, \)
and this is the general solution of
the system.
Example 2:

\[ x'_1 = 3x_1 - x_2 - x_3 \]
\[ x'_2 = -2x_1 + 3x_2 + 2x_3 \]
\[ x'_3 = 4x_1 - x_2 - 2x_3 \]

Vector/matrix form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}' =
\begin{pmatrix}
  3 & -1 & -1 \\
  -2 & 3 & 2 \\
  -4 & -1 & -2
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

or

\[
x' =
\begin{pmatrix}
  3 & -1 & -1 \\
  -2 & 3 & 2 \\
  -4 & -1 & -2
\end{pmatrix}x
\]
\[ x(t) = \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} \]

is a solution.
In fact,

\[ x(t) = C_1 e^{3t} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix} + C_3 \begin{pmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{pmatrix} \]

is a solution for any numbers \( C_1, C_2, C_3 \), and this is the general solution of the system.
THEOREM: The initial-value problem

\[ x' = A(t)x + b(t), \quad x(t_0) = c \]

has a unique solution \( x = x(t) \).
II. Homogeneous Systems: General Theory: (see Section 3.2)

\[
\begin{align*}
    x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t) \\
    x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t) \\
    &\vdots \quad \vdots \\
    x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t)
\end{align*}
\]

\[x' = A(t)x. \quad (H)\]

**Note:** The zero vector \( z(t) \equiv 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \) is a solution of (H). This solution is called the **trivial solution**.
**THEOREM:** If $v_1$ and $v_2$ are solutions of (H), then $u = v_1 + v_2$ is also a solution of (H); the sum of any two solutions of (H) is a solution of (H).

**THEOREM:** If $v$ is a solution of (H) and $\alpha$ is any real number, then $u = \alpha v$ is also a solution of (H); any constant multiple of a solution of (H) is a solution of (H).
In general,

**THEOREM:** If $v_1, v_2, \ldots, v_k$ are solutions of (H), and if $c_1, c_2, \ldots, c_k$ are real numbers, then

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).
Linear Dependence/Independence
– in general

Let

\[ \mathbf{v}_1(t) = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix}, \quad \mathbf{v}_2(t) = \begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{pmatrix}, \]

\[ \ldots, \quad \mathbf{v}_k(t) = \begin{pmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{pmatrix} \]

be vector functions defined on some interval \( I \).
The vectors are **linearly dependent** on $I$ if there exist $k$ real numbers $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1v_1(t)+c_2v_2(t)+\cdots+c_kv_k(t) \equiv 0 \quad \text{on } I.$$ 

Otherwise the vectors are **linearly independent** on $I$. 
THEOREM Let

$$v_1(t), v_2(t), \ldots, v_k(t)$$

be $k$, $k$-component vector functions defined on an interval $I$. If the vectors are linearly dependent, then

$$\begin{vmatrix}
 v_{11} & v_{12} & \cdots & v_{1k} \\
 v_{21} & v_{22} & \cdots & v_{2k} \\
 \vdots & \vdots & \ddots & \vdots \\
 v_{k1} & v_{k2} & \cdots & v_{kk}
\end{vmatrix} \equiv 0 \quad \text{on} \quad I.$$
The determinant

\[
\begin{vmatrix}
  v_{11} & v_{12} & \cdots & v_{1k} \\
v_{21} & v_{22} & \cdots & v_{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
v_{k1} & v_{k2} & \cdots & v_{kk}
\end{vmatrix}
\]

is called the **Wronskian** of the vector functions \( v_1, v_2, \ldots, v_k \).
Special case: \( n \) solutions of (H)

**THEOREM** Let \( v_1, v_2, \ldots, v_n \) be \( n \) solutions of (H). Exactly one of the following holds:

1. \( W(v_1, v_2, \ldots, v_n)(t) \equiv 0 \) on \( I \) and the solutions are linearly dependent.

2. \( W(v_1, v_2, \ldots, v_n)(t) \neq 0 \) for all \( t \in I \) and the solutions are linearly independent.
**THEOREM** Let \( v_1, v_2, \ldots, v_n \) be \( n \) linearly independent solutions of (H). Let \( u \) be any solution of (H). Then there exists a unique set of constants \( c_1, c_2, \ldots, c_n \) such that

\[
u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.
\]

That is, every solution of (H) can be written as a unique linear combination of \( v_1, v_2, \ldots, v_n \).
A set of \( n \) linearly independent solutions of (H)\\
\[ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \]
is called a fundamental set of solutions. A fundamental set is also called a solution basis for (H).

The matrix whose columns are\\
\[ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \]
is called a fundamental matrix.
Let \( v_1, v_2, \ldots, v_n \) be a fundamental set of solutions of \((H)\). Then

\[ x = C_1v_1 + C_2v_2 + \cdots + C_nv_n, \]

\( C_1, C_2, \ldots, C_n \) arbitrary constants, is the general solution of \((H)\).
A \( n^{th} \) linear equation can be converted into a system of \( n \) first order linear equations

Consider the second order equation

\[ y'' + p(t)y' + q(t)y \]

Solve for \( y'' \)

\[ y'' = -q(t)y - p(t)y' \]
Introduce new dependent variables $x_1$, $x_2$, as follows:

\[
x_1 = y
\]

\[
x_2 = x'_1 \quad (= y')
\]
Vector-matrix form:

$$
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}' =
\begin{pmatrix}
    0 & 1 \\
    -q & -p
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
$$

Note that this system is just a very special case of the “general” homogeneous system of two, first-order differential equations:

$$
x_1' = a_{11}(t)x_1 + a_{12}(t)x_2
$$

$$
x_2' = a_{21}(t)x_1 + a_{22}(t)x_2
$$
Vector-matrix form:

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) \\
  a_{21}(t) & a_{22}(t)
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

or

\[x' = A(t)x\]
Example 1:  \[ y'' - 5y' + 6y = 0 \]

Characteristic equation:

Fundamental set:

General solution:
In system form:

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

Corresponding solutions of system????

Solution of equation \( y \)

Corresponding solution of system

\[
x = \begin{pmatrix} y \\ y' \end{pmatrix}
\]

\[
e^{2t} \rightarrow \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \quad e^{3t} \rightarrow \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}
\]
\[ x_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \quad x_2 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} \]

is a fundamental set of solutions of the system

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

The matrix

\[ X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} \]

is a fundamental matrix.
Example 2: \[ y'' - \frac{5}{x} y' + \frac{8}{x^2} y = 0 \]

Look for solutions of the form \( y = t^r \)

\( y_1 = t^2, \ y_2 = t^4 \) are independent solutions
In system form:

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -8/x^2 & 5/x \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

Corresponding solutions of system????

Solution of equation \( y \)

Corresponding solution of system

\[
x = \begin{pmatrix} y \\ y' \end{pmatrix}
\]

\[
t^2 \rightarrow \begin{pmatrix} t^2 \\ 2t \end{pmatrix}, \quad t^4 \rightarrow \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix}
\]
\[ x_1 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}, \quad x_2 = \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix} \]

is a fundamental set of solutions of the system

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -8/x^2 & 5/x \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

The matrix

\[ X(t) = \begin{pmatrix} t^2 & t^4 \\ 2t & 4t^3 \end{pmatrix} \]

is a fundamental matrix.
Consider the third-order equation

\[ y''' + p(t)y'' + q(t)y' + r(t)y = 0 \]

or

\[ y''' = -r(t)y - q(t)y' - p(t)y''. \]
Introduce new dependent variables $x_1, x_2, x_3$, as follows:

\[ x_1 = y \]
\[ x_2 = x_1' \quad (= y') \]
\[ x_3 = x_2' \quad (= y'') \]

Then

\[ y''' = x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3 \]

The third-order equation can be written equivalently as the system of three first-order equations:
\[ x'_{1} = x_{2} \]
\[ x'_{2} = x_{3} \]
\[ x'_{3} = -r(t)x_{1} - q(t)x_{2} - p(t)x_{3} \]

That is

\[ x'_{1} = 0x_{1} + 1x_{2} + 0x_{3} \]
\[ x'_{2} = 0x_{1} + 0x_{2} + 1x_{3} \]
\[ x'_{3} = -r(t)x_{1} - q(t)x_{2} - p(t)x_{3} \]
Vector-matrix form:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}' = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-r & -q & -p \\
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}
\]
Note that this system is just a very special case of the “general” system of three, first-order differential equations:

\[ x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3(t) \]
\[ x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3(t) \]
\[ x'_3 = a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3(t) \]

or in vector-matrix form:

\[ \mathbf{x}' = A(t)\mathbf{x} \]
Example 3:

\[ y''' - 3y'' - 4y' + 12y = 0. \]

which can be written

\[ y''' = -12y + 4y' + 3y''. \]

Set

\[ x_1 = y \]
\[ x_2 = x_1' (= y') \]
\[ x_3 = x_2' (= y'') \]
Then

\[ x_3' = y''' = -12x_1 + 4x_2 + 3x_3 \]

and equivalent system:

\[ x_1' = x_2 \]
\[ x_2' = x_3 \]
\[ x_3' = -12x_1 + 4x_2 + 3x_3 \]

which is

\[ x_1' = 0x_1 + 1x_2 + 0x_3 \]
\[ x_2' = 0x_1 + 0x_2 + 1x_3 \]
\[ x_3' = -12x_1 + 4x_2 + 3x_3 \]
Vector-matrix form:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \\
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\
\end{pmatrix}
\]

or

\[x' = Ax\]
\[ y''' - 3y'' - 4y' + 12y = 0 \]

Characteristic equation:

\[ r^3 - 3r^2 - 4r + 12 = (r - 3)(r - 2)(r + 2) \]

Fundamental set:

\[ \{ e^{3t}, e^{2t}, e^{-2t} \} \]

General solution:

\[ y = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t} \]
\[ y = e^{3t} \] is a solution of the equation.

System:

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}' = \begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
   -12 & 4 & 3
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
\]

Recall:

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix} = \begin{pmatrix}
    y \\
    y' \\
    y''
\end{pmatrix}
\]

\[
x = \begin{pmatrix}
    y \\
    y' \\
    y''
\end{pmatrix} = \begin{pmatrix}
    e^{3t} \\
    3e^{3t} \\
    9e^{3t}
\end{pmatrix}
\]

is a corresponding solution of the system.
Equation:

\[ y''' - 3y'' - 4y' + 12y = 0 \]

Fundamental set:

\[ \{ y_1 = e^{3t}, \quad y_2 = e^{2t}, \quad y_3 = e^{-2t} \} \]

Equivalent vector-matrix system:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}' =
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -12 & 4 & 3
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

Solutions:

\[ y_1 = e^{3t} \longrightarrow x_1 = \begin{pmatrix}
  e^{3t} \\
  3e^{3t} \\
  9e^{3t}
\end{pmatrix} \]
\[ y_2 = e^{2t} \quad \rightarrow \quad x_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} \]

\[ y_3 = e^{-2t} \quad \rightarrow \quad x_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} \]
\[
x_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}, \quad x_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}
\]

\[
x_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix}
\]

is a fundamental set of solutions of
the corresponding system

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]
III. Homogeneous Systems with Constant Coefficients (see Section 3.3)

\[
\begin{align*}
    x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
    x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
    \quad &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
The system in vector-matrix form is

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}' =
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

or

\[ x' = Ax. \]
Solutions of $x' = Ax$:

Example 1. See page 27

$$x_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is a solution of

$$x' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} x$$

How is the number 2 and the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ related to the matrix $\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$?
\[
\begin{pmatrix}
0 & 1 \\
-6 & 5
\end{pmatrix}
\begin{pmatrix}
1 \\
2
\end{pmatrix}
= 
\]
NOTE:

\[ y'' - 5y' + 6y = 0 \]

Characteristic equation

\[ r^2 - 5r + 6 = (r - 2)(r - 3) = 0 \]

Characteristic roots: \( r_1 = 2, \ r_2 = 3 \)

Fundamental set:

\[ \{ y_1 = e^{2t}, \quad y_2 = e^{3t} \} \]
Vector-matrix system

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}' = \begin{pmatrix}
    0 & 1 \\
    -6 & 5
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

\[A = \begin{pmatrix}
    0 & 1 \\
    -6 & 5
\end{pmatrix}\]

Characteristic equation:

\[
\det (A - \lambda I) = \begin{vmatrix}
    -\lambda & 1 \\
    -6 & 5 - \lambda
\end{vmatrix}
\]

\[= \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0\]

Eigenvalues: \(\lambda_1 = 2, \lambda_2 = 3\)

Fund set: \(x_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}\)
Example 2.

\[ x_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \]

is a solution of

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \]

How is the number 3 and the vector \[ \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \] related to the matrix

\[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \]
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-12 & 4 & 3
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
9
\end{pmatrix}
= 
\]
THAT IS:

3 is an eigenvalue of $A$ and

$$v = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a corresponding eigenvector.
\[ y'''' - 3y'' - 4y' + 12y = 0 \]

Characteristic equation:

\[ r^3 - 3r^2 - 4r + 12 = (r-3)(r-2)(r+2) = 0 \]

Characteristic roots:

\[ r_1 = 3, \quad r_2 = 2, \quad r_3 = -2 \]

Fundamental set:

\[ \{ y_1 = e^{3t}, \quad y_2 = e^{2t}, \quad y_3 = e^{-2t} \} \]
Vector-matrix form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -12 & 4 & 3
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

Characteristic equation:

\[
\det (A - \lambda I) = \begin{vmatrix}
  -\lambda & 1 & 0 \\
  0 & -\lambda & 1 \\
  -12 & 4 & 3 - \lambda
\end{vmatrix}
\]

\[
= -\lambda^3 + 3\lambda^2 + 4\lambda - 12 = 0
\]

or

\[
\lambda^3 - 3\lambda^2 - 4\lambda + 12 = (\lambda - 3)(\lambda - 2)(\lambda + 2) = 0
\]
Eigenvalues:

\[ \lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = -2 \]

Eigenveectors:

\[
y_1 = e^{3t} \quad \rightarrow \quad x_1 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}
\]

\[
y_2 = e^{2t} \quad \rightarrow \quad x_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}
\]

\[
y_3 = e^{-2t} \quad \rightarrow \quad x_3 = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}
\]
Given the homogeneous system with constant coefficients

\[ x' = Ax. \]

**THEOREM:** If \( \lambda \) is an eigenvalue of \( A \) and \( v \) is a corresponding eigenvector, then

\[ x = e^{\lambda t}v \]

is a solution.
Proof:

Let $\lambda$ be an eigenvalue of $A$ with corresponding eigenvector $v$.

Set $x = e^{\lambda t}v$
If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of $A$ with corresponding eigenvectors $v_1, v_2, \cdots, v_k$, then

$$x_1 = e^{\lambda_1 t}v_1, \ x_2 = e^{\lambda_2 t}v_2, \cdots, x_k = e^{\lambda_k t}v_k$$

are linearly independent solutions of

$$x' = Ax.$$
In particular: If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigenvalues of $A$ with corresponding eigenvectors $v_1, v_2, \cdots, v_n$, then

$$x_1 = e^{\lambda_1 t}v_1, \; x_2 = e^{\lambda_2 t}v_2, \; \cdots, \; x_n = e^{\lambda_k t}v_n$$

form a fundamental set of solutions of

$$x' = Ax.$$ 

and

$$x = C_1x_1 + C_2x_2 + \cdots + C_nx_n$$

is the general solution.
Example 1. Find the general solution of

\[ x' = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} x. \]

Step 1. Find the eigenvalues of \( A \):

\[
\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 6.
\]

Characteristic equation:

\[
\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.
\]

Eigenvalues: \( \lambda_1 = 3, \lambda_2 = -2 \).
Step 2. Find the eigenvectors:

\[ A - \lambda I = \begin{pmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{pmatrix} \]

\( \lambda_1 = 3: \)
\[ A - \lambda I = \begin{pmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{pmatrix} \]

\[ \lambda_2 = -2 \]
\[ \lambda_1 = 3, \ v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \ \lambda_2 = -2, \ v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \]

**Solutions:**

Fundamental set of solution vectors:

\[ \begin{cases} x_1 = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, & x_2 = e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{cases} \]

General solution of the system:

\[ x = C_1 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \]
Example 2. Solve \( x' = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix} x \).

**Step 1.** Find the eigenvalues of \( A \):

\[
\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1 - \lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2.
\]

Characteristic equation:

\[
\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 2)(\lambda - 1)(\lambda + 1) = 0.
\]

Eigenvalues:

\[
\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1.
\]
Step 2. Find the eigenvectors:

\[ A - \lambda I = \begin{pmatrix} 3 - \lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1 - \lambda \end{pmatrix} \]

\[ \lambda_1 = 2: \]
$$\lambda_1 = 2 : \quad v_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

$$\lambda_2 = 1 : \quad v_2 = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix},$$

$$\lambda_3 = -1 : \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Fundamental set of solutions:

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x_2 = e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix},$$

$$x_3 = e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$
The general solution of the system:

\[ x = C_1 e^{2t} \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]
Example 3. Solve the initial-value problem

\[ x' = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \]

To find the solution vector satisfying the initial condition, solve

\[ C_1 v_1(0) + C_2 v_2(0) + C_3 v_3(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]

which is:

\[ C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 7 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]
or

$$
\begin{pmatrix}
1 & 3 & 1 \\
-1 & -1 & 2 \\
2 & 7 & 2
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
C_3
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}.
$$

Augmented matrix:

$$
\begin{pmatrix}
1 & 3 & 1 & 1 \\
-1 & -1 & 2 & 0 \\
2 & 7 & 2 & 1
\end{pmatrix}
$$
Solution:

\[ C_1 = 3, \quad C_2 = -1, \quad C_3 = 1. \]

The solution of the initial-value problem is:

\[ x = 3e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}. \]
TWO DIFFICULTIES:

1. $A$ has complex eigenvalues and complex eigenvectors.

2. $A$ has an eigenvalue of multiplicity greater than 1.
1. Complex eigenvalues/eigenvectors

Example 1. Find the general solution of

$$x' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} x.$$ 

$$\text{det}(A-\lambda I) = \begin{vmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5.$$ 

The eigenvalues are:

$$\lambda_1 = -1 + 2i, \; \lambda_2 = -1 - 2i.$$
\[ A - \lambda I = \begin{pmatrix} -3 - \lambda & -2 \\
4 & 1 - \lambda \end{pmatrix} \]

For \( \lambda_1 = -1 + 2i \): Solve

\[
\begin{pmatrix} -2 - 2i & -2 \\
4 & 2 - 2i \end{pmatrix} \rightarrow
\]

\[ \begin{pmatrix} 0 \\
0 \end{pmatrix} \]
The solution set is:

\[ x_2 = -(1 + i)x_1, \quad x_1 \text{ arbitrary} \]

Set \( x_1 = 1 \). Then, for \( \lambda_1 = -1 + 2i \):

\[ v_1 = \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \]

and, for \( \lambda_2 = -1 - 2i \):

\[ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \]
Solutions

\[ u_1 = e^{\lambda_1 t} v_1 = \]

\[ = e^{(-1+2i)t} \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] = \]

\[ = e^{-t}(\cos 2t + i \sin 2t) \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \]

\[ = e^{-t} \left[ \cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + \]

\[ i e^{-t} \left[ \cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]. \]
\[ u_2 = e^{\lambda_2 t} v_2 \]

\[ = e^{(-1-2i)t} \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] = \]

\[ = e^{-t}(\cos 2t + i \sin 2t) \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \]

\[ = e^{-t} \left[ \cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] - \]

\[ i e^{-t} \left[ \cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] . \]
Fundamental set:

\[ x_1 = \frac{u_1 + u_2}{2} = e^{-t} \begin{bmatrix} \cos 2t \left( \begin{array}{c} 1 \\ -1 \end{array} \right) - \sin 2t \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \end{bmatrix} \]

\[ x_2 = \frac{u_1 + u_2}{2i} = e^{-t} \begin{bmatrix} \cos 2t \left( \begin{array}{c} 0 \\ -1 \end{array} \right) + \sin 2t \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \end{bmatrix} \]

General solution:

\[ x = C_1 e^{-t} \begin{bmatrix} \cos 2t \left( \begin{array}{c} 1 \\ -1 \end{array} \right) - \sin 2t \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \cos 2t \left( \begin{array}{c} 0 \\ -1 \end{array} \right) + \sin 2t \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \end{bmatrix} \]
Graphs
Example 2. Find the general solution of

\[ x' = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} x. \]

\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -5 \\ 2 & 3 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13.
\]

Characteristic equation:

\[ \lambda^2 - 4\lambda + 13 = 0 \]

Eigenvalues:

\[ \lambda_1 = 2 + 3i, \quad \lambda_2 = 2 - 3i. \]
\[ A - \lambda I = \begin{pmatrix} 1 - \lambda & -5 \\ 2 & 3 - \lambda \end{pmatrix} \]

For \( \lambda_1 = 2 + 3i \): Solve

\[
\begin{pmatrix} -1 - 3i & -5 & 0 \\ 2 & 1 - 3i & 0 \end{pmatrix} \rightarrow
\]
Eigenvectors:

\[ \lambda_1 = 2 + 3i, \quad v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}. \]

\[ \lambda_2 = 1 - 3i, \quad v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 0 \end{pmatrix}. \]

General solution:

\[ x = C_1 e^t \left[ \cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right] + 
C_2 e^t \left[ \cos 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right] \]
Graphs
Summary: $x' = Ax$, $A \in \mathbb{R}^{n \times n}$ constant.

$a + ib, a - ib$ complex eigenvalues.

$\vec{\alpha} + i \vec{\beta}, \vec{\alpha} - i \vec{\beta}$ corresponding eigenvectors.

Independent (complex-valued) solutions:

$$u_1 = e^{(a+ib)t} \left( \vec{\alpha} + i \vec{\beta} \right)$$

$$u_2 = e^{(a-ib)t} \left( \vec{\alpha} - i \vec{\beta} \right)$$
Corresponding real-valued solutions:

\[ x_1 = e^{at} \left[ \cos bt \overrightarrow{\alpha} - \sin bt \overrightarrow{\beta} \right] \]

\[ x_2 = e^{at} \left[ \cos bt \overrightarrow{\beta} + \sin bt \overrightarrow{\alpha} \right] \]

General solution:

\[ x = C_1 e^{at} \left[ \cos bt \overrightarrow{\alpha} - \sin bt \overrightarrow{\beta} \right] + C_2 e^{at} \left[ \cos bt \overrightarrow{\beta} + \sin bt \overrightarrow{\alpha} \right] \]
Example 3. Determine a fundamental set of solution vectors of

$$x' = \begin{pmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} x.$$

$$\det(A - \lambda I) \equiv \begin{vmatrix} 1 - \lambda & -4 & -1 \\ 3 & 2 - \lambda & 3 \\ 1 & 1 & 3 - \lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 21\lambda + 26 = -(\lambda - 2)(\lambda^2 - 4\lambda + 13).$$

The eigenvalues are:

$$\lambda_1 = 2, \ \lambda_2 = 2 + 3i, \ \lambda_3 = 2 - 3i.$$
\[ A - \lambda I = \begin{pmatrix} 1 - \lambda & -4 & -1 \\ 3 & 2 - \lambda & 3 \\ 1 & 1 & 3 - \lambda \end{pmatrix} \]

\[ \lambda_1 = 2: \text{ Solve} \]

\[
\begin{pmatrix} -1 & -4 & -1 & 0 \\ 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow
\]

\[ v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]
\[ A - \lambda I = \begin{pmatrix}
1 - \lambda & -4 & -1 \\
3 & 2 - \lambda & 3 \\
1 & 1 & 3 - \lambda
\end{pmatrix} \]

For \( \lambda_2 = 2 + 3i \): Solve

\[ \begin{pmatrix}
-1 - 3i & -4 & -1 \\
3 & -3i & 3 \\
1 & 1 & 1 - 3i
\end{pmatrix} \rightarrow \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \]
The solution set is:

\[ x_1 = \left( -\frac{5}{2} + \frac{3}{2}i \right) x_3, \quad x_2 = \left( \frac{3}{2} + \frac{3}{2}i \right) x_3, \]

\[ x_3 \text{ arbitrary.} \]

\[ v_2 = \begin{pmatrix} -5 + 3i \\ 3 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}. \]

and

\[ v_3 = \begin{pmatrix} -5 - 3i \\ 3 - 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}. \]
Now

\[ u_1 = e^{(2+3i)t} \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \]

and

\[ u_2 = e^{(2-3i)t} \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \]

can be converted to:

\[ x_1 = e^{2t} \begin{pmatrix} \cos 3t & \cos 3t & \cos 3t \end{pmatrix} \]

\[ \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \]

\[ - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \]

and

\[ x_2 = e^{2t} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \]
Fundamental set of solution vectors:

\[ x_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \]

\[ x_2 = e^{2t} \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \cos 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \]

\[ x_3 = e^{2t} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}. \]

General solution:

\[ x = C_1 x_1 + C_2 x_2 + C_3 x_3 \]
2. Repeated eigenvalues

Example 1. Find a fundamental set of solutions of

\[ x' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} x. \]

\[ \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{vmatrix} \]

\[ = 16 + 12\lambda - \lambda^3 = -(\lambda - 4)(\lambda + 2)^2. \]

Eigenvalues: \( \lambda_1 = 4, \lambda_2 = \lambda_3 = -2 \)
\[ \lambda_1 = 4 : \quad (A - 4I) = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \]

Solve:

\[ (A - 4I)x = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
\( \lambda_2 = \lambda_3 = -2: \)

\[
A - (-2)I = \begin{pmatrix}
3 & -3 & 3 \\
3 & -3 & 3 \\
6 & -6 & 6
\end{pmatrix}
\]
which row reduces to

\[
\begin{pmatrix}
1 & -1 & 1 & | & 0 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{pmatrix}.
\]

Solution set:

\[
x_1 = a - b, \quad x_2 = a, \quad x_3 = b
\]

\(a, b\) any real numbers.

Set \(a = 1, b = 0\) : \(v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\);

Set \(a = 0, b = -1\) : \(v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\).
Fundamental set:

\[
\left\{ e^{4t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.
\]
**Example 2.** Find a fundamental set of solutions of \( x' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}x \).

\[
\det(A - \lambda I) = \begin{vmatrix} -4 - \lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4.
\]

Characteristic equation:

\[
\lambda^2 + 4\lambda + 4 = 0
\]

Eigenvalues:

\[
\lambda_1 = \lambda_2 = -2.
\]
Eigenvectors: \( A - \lambda I = \begin{pmatrix} -4 - \lambda & 1 \\ -4 & -\lambda \end{pmatrix} \)

\( \lambda_1 = \lambda_2 = -2 \): Solve

\[(A - (-2)I)x = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[
\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \rightarrow
\]

**Problem:** Only one eigenvector and only one solution! We need another solution.
Graphs
Example 3. Find a fundamental set of solutions of

\[
x' = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix} x.
\]

\[
\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & 6 & 2 \\ 0 & -1 - \lambda & -8 \\ 1 & 0 & -2 - \lambda \end{vmatrix}
= -36 + 15\lambda + 2\lambda^2 - \lambda^3 = -(\lambda + 4)(\lambda - 3)^2.
\]

Eigenvalues: \(\lambda_1 = -4, \lambda_2 = \lambda_3 = 3.\)
\[ \lambda_1 = -4: \quad A - (-4)I = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 3 & -8 \\ 1 & 0 & 2 \end{pmatrix} \]

\[
\begin{pmatrix} 9 & 6 & 2 & | & 0 \\ 0 & 3 & -8 & | & 0 \\ 1 & 0 & 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 3 & -8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \rightarrow 
\]

\[
x_1 = -2x_3, \quad x_2 = \frac{8}{3}x_3, \quad x_3 \text{ arbitrary} 
\]

Set \( x_3 = -3 \) : \quad \mathbf{v}_1 = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}

\[
x_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}
\]
$\lambda_2 = \lambda_3 = 3:$

$$A - 3I = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}$$

Solve

$$(A - 3I)x = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 6 & 2 & | & 0 \\ 0 & -4 & -8 & | & 0 \\ 1 & 0 & -5 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5 & | & 0 \\ 0 & -4 & -8 & | & 0 \\ 2 & 6 & 2 & | & 0 \end{pmatrix}$$
\[
\begin{pmatrix}
2 & 6 & 2 & | & 0 \\
0 & -4 & -8 & | & 0 \\
1 & 0 & -5 & | & 0 \\
\end{pmatrix}
\]
which row reduces to
\[
\begin{pmatrix}
1 & 0 & -5 & | & 0 \\
0 & 1 & 2 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{pmatrix}
\].

\[x_1 = 5x_3, \ x_2 = -2x_3, \ x_3 \ \text{arbitrary}\]

Set \[x_3 = 1: \ v_2 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}\]

**Problem:** Only one eigenvector here!
Solutions:

\[ x_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad x_2 = e^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}. \]

Problem: Only two solutions!

We need a third solution \( x_3 \) which is independent of \( x_1, x_2 \).
Consider the linear equation

\[ y''' + y'' - 8y' - 12y = 0 \]

Char.eqn.

\[ r^3 + r^2 - 8r - 12 = (r - 3)(r + 2)^2 = 0. \]

Fundamental set: \( \{ e^{3t}, e^{-2t}, te^{-2t} \} \)
Equivalent system: \( x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 8 & -1 \end{pmatrix} x \)

\[
\text{det}(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 12 & 8 & -1 - \lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 8\lambda + 12\lambda
\]

char. eqn.:

\[
\lambda^3 + \lambda^2 - 8\lambda - 12 = (\lambda - 3)(\lambda + 2)^2
\]

Eigenvalues: \( \lambda_1 = 3, \quad \lambda_2 = \lambda_3 = -2 \)
Fundamental set:

\[ x_1 = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad x_2 = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \]

\[ x_3 = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \]

Question:

What is the vector \( \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \) ???
\[
[A - (-2)I] \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = 109
\]
$A - (-2I)$ “maps” \[
\begin{pmatrix}
0 \\
1 \\
-4
\end{pmatrix}
\] onto the eigenvector \[
\begin{pmatrix}
1 \\
-2 \\
4
\end{pmatrix}
\].

$w = \begin{pmatrix}
0 \\
1 \\
-4
\end{pmatrix}$

is called a generalized eigenvector.

The third solution has the form

$$x_3 = e^{-2t}w + te^{-2t}v$$
Back to Example 3. The third solution has the form

\[ x_3 = e^{3t}w + te^{3t}v \]

Solve

\[
(A - 3I)w = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 2 & 6 & 2 & | & 5 \\ 0 & -4 & -8 & | & -2 \\ 1 & 0 & -5 & | & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5 & | & 1 \\ 0 & -4 & -8 & | & -2 \\ 2 & 6 & 2 & | & 5 \end{pmatrix} \rightarrow
\]
\[
\begin{pmatrix}
1 & 0 & -5 & 1 \\
0 & 1 & 2 & 1/2 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Solution set:

\[w_1 = 1 + 5w_3, \ w_2 = \frac{1}{2} - 2w_3, \ w_3 \text{ arbitrary}\]

Set \( w_3 = 0 \):

\[w = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}\]

\[x_3 = e^{3t} \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}\]
Fundamental set:

\[ x_1 = e^{-4t} \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad x_2 = e^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \]

\[ x_3 = e^{3t} \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + te^{3t} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \]
Back to Example 2. Solve

\[(A - (-2)I)w = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\]
Fundamental set:

\[ x_1 = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \]

\[ x_2 = e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]
Eigenvalues of multiplicity 2:

Given \( \mathbf{x}' = A \mathbf{x} \).

Suppose that \( A \) has an eigenvalue \( \lambda \) of multiplicity 2. Then exactly one of the following holds:
1. \( \lambda \) has two linearly independent eigenvectors, \( v_1 \) and \( v_2 \). Corresponding linearly independent solution vectors of the differential system are

\[
x_1(t) = e^{\lambda t}v_1 \quad \text{and} \quad x_2(t) = e^{\lambda t}v_2.
\]

(See Example 1.)
2. $\lambda$ has only one eigenvector $v$. (See Examples 2 and 3.) Then a linearly independent pair of solution vectors corresponding to $\lambda$ is:

$$x_1(t) = e^{\lambda t}v \quad \text{and} \quad x_2(t) = e^{\lambda t}w + te^{\lambda t}v$$

where $w$ is a vector that satisfies

$$(A - \lambda I)w = v.$$ 

The vector $w$ is called a **generalized eigenvector** corresponding to the eigenvalue $\lambda$. 
**Examples:** Find a fundamental set of solutions and the general solution.

1. \[ x' = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix} x. \]

\[
\text{det} \left( A - \lambda I \right) = \begin{vmatrix} 2 - \lambda & 5 \\ -1 & 4 - \lambda \end{vmatrix} \\
= \lambda^2 - 6\lambda + 13
\]

Eigenvalues: \( 3 + 2i, \quad 3 - 2i \)
\[(A - \lambda I) = \begin{pmatrix} 2 - \lambda & 5 \\ -1 & 4 - \lambda \end{pmatrix}\]

\[\lambda_1 = 3 + 2i, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -2 \\ 0 \end{pmatrix}\]
Fundamental set:

\[ x_1 = e^{3t} \left[ \cos 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right] \]

\[ x_2 = e^{3t} \left[ \cos 2t \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \]

General solution:

\[ x(t) = C_1 x_1 + C_2 x_2 \]
2. \[ x' = \begin{pmatrix} -4 & 1 & -2 \\ 2 & -3 & 2 \\ 2 & -1 & 0 \end{pmatrix} x. \]

HINT: \(-3\) is an eigenvalue and \(-2\) is an eigenvalue of multiplicity 2

Characteristic eqn: \((\lambda+3)(\lambda+2)^2=0\)
\[(A - \lambda I) = \begin{pmatrix} -4 - \lambda & 1 & -2 \\ 2 & -3 - \lambda & 2 \\ 2 & -1 & -\lambda \end{pmatrix} \]

\[\lambda_1 = -3: \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\]
\[(A - \lambda I) = \begin{pmatrix}
-4 - \lambda & 1 & -2 \\
2 & -3 - \lambda & 2 \\
2 & -1 & -\lambda 
\end{pmatrix}
\]

\[
\lambda_2 = \lambda_3 = -2 : \quad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix}
-2 & 1 & -2 | 0 \\
2 & -1 & 2 | 0 \\
2 & -1 & 2 | 0 
\end{pmatrix} \rightarrow
\]
Fundamental set:

\[ e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad e^{-2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \]

General solution:

\[ x = C_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \]
3. \[ x' = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix} x. \]

HINT: 4 is an eigenvalue and \(-2\) is an eigenvalue of multiplicity 2

Characteristic eqn: \((\lambda - 4)(\lambda + 2)^2 = 0\)
\[(A - \lambda I) = \begin{pmatrix}
-3 - \lambda & 1 & -1 \\
-7 & 5 - \lambda & -1 \\
-6 & 6 & -2 - \lambda
\end{pmatrix}\]

\[\lambda_1 = 4 : \quad v_1 = \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}\]
\[(A - \lambda I) = \begin{pmatrix} -3 - \lambda & 1 & -1 \\ -7 & 5 - \lambda & -1 \\ -6 & 6 & -2 - \lambda \end{pmatrix} \]

\[\lambda_2 = \lambda_3 = -2: \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\]
\[(A - \lambda I) = \begin{pmatrix}
-3 - \lambda & 1 & -1 \\
-7 & 5 - \lambda & -1 \\
-6 & 6 & -2 - \lambda
\end{pmatrix}\]

\[[A - (-2)I]w = \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}\]
Fund. Set: \( e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \)

\[ e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]

General solution:

\[ \mathbf{x} = C_1 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \]

\[ C_3 \begin{bmatrix} e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{bmatrix} \]
4. \[ x' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} x \]

Characteristic polynomial:

\[
\det (A - \lambda I) = \begin{vmatrix} 1 - \lambda & -2 \\ 2 & 5 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9
\]

Characteristic equation:

\[ \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \]

Eigenvalues: \[ \lambda_1 = \lambda_2 = 3 \]
Eigenvectors:

\[(A - 3I)x = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\]

\[
\begin{pmatrix} -2 & -2 & | & 0 \\ 2 & 2 & | & 0 \end{pmatrix} \rightarrow
\]
\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[
\begin{pmatrix}
-2 & -2 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix} \rightarrow
\]
Fundamental set:

\[ x_1 = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ x_2 = e^{3t} \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + te^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

General solution:

\[ x = C_1 e^{3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \\
C_2 \left[ e^{3t} \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + te^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \]
Graphs
5. \[ x' = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} x. \]

\[
\det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{vmatrix}
\]

\[= \lambda^2 + 2\lambda + 5.\]

Characteristic equation:

\[\lambda^2 + 2\lambda + 5 = 0\]

Eigenvalues:

\[\lambda_1 = -1 + 2i, \quad \lambda_2 = -1 - 2i.\]
Eigenvectors:

\[ \lambda_1 = -1 + 2i, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \]

\[ \lambda_2 = -1 - 2i, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \]

General solution:

\[ x = C_1 e^{-t} \left[ \cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + \]

\[ C_2 e^{-t} \left[ \cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \]
Graphs
Given the differential system

\[ x' = Ax. \]

Suppose that \( \lambda \) is an eigenvalue of \( A \) of multiplicity 3. Then exactly one of the following holds:
1. \( \lambda \) has three linearly independent eigenvectors \( v_1, v_2, v_3 \). Then three linearly independent solution vectors of the system corresponding to \( \lambda \) are:

\[
x_1(t) = e^{\lambda t}v_1, \quad x_2(t) = e^{\lambda t}v_2, \quad x_3(t) = e^{\lambda t}v_3.
\]
2. $\lambda$ has two linearly independent eigenvectors $v_1, v_2$. Then three linearly independent solutions of the system corresponding to $\lambda$ are:

$$x_1(t) = e^{\lambda t} v_1, \quad x_2(t) = e^{\lambda t} v_2$$

and

$$x_3(t) = e^{\lambda t} w + t e^{\lambda t} v$$

where $v$ is an eigenvector corresponding to $\lambda$ and $(A - \lambda I)w = v$. That is: $(A - \lambda I)^2 w = 0$. 

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3. $\lambda$ has only one (independent) eigenvector $v$. Then three linearly independent solutions of the system have the form:

$$x_1 = e^{\lambda t}v, \quad x_2 = e^{\lambda t}w + te^{\lambda t}v,$$

$$v_3(t) = e^{\lambda t}z + te^{\lambda t}w + t^2e^{\lambda t}v$$

where

$$(A - \lambda I)z = w \quad \& \quad (A - \lambda I)w = v, \quad i.e.$$

$$(A - \lambda I)^3z = 0 \quad \& \quad (A - \lambda I)^2w = 0$$
Example:

\[ y''' - 6y'' + 12y' - 8y = 0 \]

Char. eqn.: \( (r - 2)^3 = 0 \)

Char. roots: \( r_1 = r_2 = r_3 = 2 \)

Fundamental set:

\( \{ e^{2t}, \ t e^{2t}, \ t^2 e^{2t} \} \)
Corresponding system:

\[
x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{pmatrix} x
\]

Fundamental set:

\[
x_1 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad x_2 = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},
\]

\[
x_3 = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + te^{2t} \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix} + t^2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}
\]