A differential equation is an equation that contains an unknown function together with one or more of its derivatives.
Examples:

1. \( y' = 2x + \cos x \)

2. \( \frac{dy}{dt} = ky \) (exponential growth/decay)

3. \( x^2 y'' - 2xy' + 2y = 4x^3 \)
4. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \] (Laplace’s eqn.)

5. \[ \frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} = 0 \]
TYPE:

If the unknown function depends on a single independent variable, then the equation is an **ordinary differential equation** (ODE); if the unknown function depends on more than one independent variable, then the equation is a **partial differential equation** (PDE).
ORDER:

The **order** of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.
Examples:

1. \( \frac{dy}{dt} = ky \)

2. \( x^2y'' - 2xy' + 2y = 4x^3 \)

3. \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)
4. \[ \frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} = 0 \]

5. \[ \frac{d^2y}{dx^2} + 2x \sin \left( \frac{dy}{dx} \right) + 3e^{xy} = \frac{d^3}{dx^3}(e^{2x}) \]
SOLUTION:

A solution of a differential equation is a function defined on some domain $D$ such that the equation reduces to an identity when the function is substituted into the equation.
Examples:

1. \( y' = 2x + \cos x \)
\[ 2. \quad y' = ky \]

\[ y = e^{kt} \]

Set of all solutions

\[ y = Ce^{kt}, \quad C \text{ any constant} \]
3. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

\[ y = x^2 + 2x^3 \quad \text{Solution?} \]
4. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

\[ y = 2x + x^2 \quad \text{Solution?} \]
4. \[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

\[u = \ln \sqrt{x^2 + y^2}\] Solution?
5. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u = \cos x \sinh y, \quad u = 3x - 4y \]
6. Find values of \( r \) such that \( y = e^{rx} \) is a solution of

\[ y'' - 2y' - 15y = 0. \]
7. Find values of $r$ such that $y = x^r$ is a solution of

$$x^2 y'' - 4x y' + 6y = 0.$$
From now on, all differential equations are ordinary differential equations.
Example:

Solve the differential equation:

$$y''' - 12x + 6e^{2x} = 0$$
Intuitively, to find a set of solutions of an $n$-th order differential equation we “integrate” $n$ times, with each integration step producing an arbitrary constant of integration. Thus, ”in theory,” an $n$-th order differential equation has an $n$-parameter family of solutions.
SOLVING A DIFFERENTIAL EQUATION:

To **solve** an $n$-th order differential equation means to find an $n$-parameter families of solutions. (Same $n$.)
Examples: $n$-parameter family of solutions:

1. $y' = 3x^2 - 2x + 4$

Answer: $y = x^3 - x^2 + 4x + C$
2. \[ y'' = 2x + \sin 2x \]

Answer:

\[ y = \frac{1}{3}x^3 - \frac{1}{4}\sin 2x + C_1x + C_2 \]
3. \( y''' - 3y'' + 3y' - y = 0 \)

Answer: \( y = C_1 e^x + C_2 xe^x + C_3 x^2 e^x \)
4. \( x^2 y'' - 2xy' + 2y = 4x^3 \)

\[ \text{Answer: } y = C_1 x + C_2 x^2 + 2x^3 \]
GENERAL SOLUTION/SINGULAR SOLUTIONS:

An “$n$-parameter family of solutions” is also called the general solution.

Solutions of an $n$-th order differential equation which are not included in an $n$-parameter family of solutions are called singular solutions.
Example:

\[
\frac{dy}{dx} = (4x + 2)(y - 2)^{1/3}
\]

**General solution:**

\[(y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x + C\]

**Singular solution:** \[y \equiv 2\]
PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a particular solution of the equation.
Examples:

1. \( x^2y'' - 2xy' + 2y = 4x^3 \)

General solution:

\[ y = C_1x + C_2x^2 + 2x^3 \]

Particular solutions:

\[ y_1 = 2x^3 \quad (C_1 = C_2 = 0) \]

\[ y_2 = 3x - 2x^2 + 2x^3 \quad (C_1 = 3, \ C_2 = -2) \]
\[
\frac{dy}{dx} = (4x + 2)(y - 2)^{1/3}
\]

General solution:

\[
(y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x + C
\]

Particular solutions:

\[
C = 0 : \quad (y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x
\]

\[
C = -5 : \quad (y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x - 5
\]

Singular solution: \( y \equiv 2 \)
THE DIFFERENTIAL EQUATION OF AN $n$-PARAMETER FAMILY:

Given an $n$-parameter family of curves. The differential equation of the family is an $n$-th order differential equation that has the given family as its general solution.
Examples:

1. \( y^2 = Cx^3 + 4 \)

is the general solution of a DE. What's the equation?

Answer: \( y' = \frac{3y^2 - 12}{2xy} \)
2. \( y = C_1 x + C_2 x^3 \) is the general solution of a DE. What’s the equation?

Answer: \[ x^2 y'' - 3xy' + 3y = 0 \]
Strategy for finding the differential equation

Step 1. Differentiate the family \( n \) times. This produces a system of \( n + 1 \) equations.

Step 2. Choose any \( n \) of the equations and solve for the parameters.

Step 3. Substitute the “values” for the parameters in the remaining equation.
Examples:

The given family of functions is the general solution of a differential equation.

(a) What is the order of the equation?

(b) Find the equation.
1. \[ y = C_1 e^{2x} + C_2 e^{3x} + C_3 \]

(a) 

(b)
2. \( y = C_1 \cos 3x + C_2 \sin 3x \)

(a)

(b)
$n$-th ORDER INITIAL-VALUE PROBLEMS:

1. Find a solution of

$$y' = 3x^2 + 2x + 1$$

which passes through the point $(-2, 4)$.

Answer: $y = x^3 + x^2 + x + 10$
2. \( y = C_1 \cos 3x + C_2 \sin 3x \) is the general solution of
\[
y'' + 9y = 0.
\]

a. Find a solution which satisfies
\[
y(0) = 3
\]

Answer: \( y = C_1 \sin 3x + 3 \cos 3x \) for any \( C_1 \).
b. Find a solution which satisfies

\[ y(0) = 4, \ y(\pi) = 4 \]

Answer: No solution!!
c. Find a solution which satisfies

\[ y(\pi/4) = 1, \quad y'(\pi/4) = 2 \]

**Answer:** \[ \frac{1}{3\sqrt{2}} \sin 3x - \frac{5}{3\sqrt{2}} \cos 3x \]
An \textit{n-th order initial-value problem} consists of an \textit{n-th order} differential equation

\[ F \left[ x, y, y', y'', \ldots, y^{(n)} \right] = 0 \]

together with \textit{n} (initial) conditions of the form

\[ y(c) = k_0, \ y'(c) = k_1, \ y''(c) = k_2, \ldots, \]

\[ y^{(n-1)}(c) = k_{n-1} \]

where \( c \) and \( k_0, k_1, \ldots, k_{n-1} \) are given numbers.
NOTES:

1. An $n$-th order differential equation can always be written in the form

$$F\left[x, y, y', y'', \ldots, y^{(n)}\right] = 0$$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.
Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.
Examples:

\[ y = C_1x + C_2x^3 \] is the general solution of

\[ x^2 y'' - 3xy' + 3y = 0 \]

a. Find the solution which satisfies

\[ y(1) = 2, \quad y'(1) = -4. \]
b. Find the solution that satisfies

\[ y(0) = 0, \quad y'(0) = 2. \]
c. Find the solution that satisfies

\[ y(0) = 2, \quad y'(0) = 3. \]

**Answer:** No solution.
EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem have a solution? That is, do solutions to the problem exist?

2. If a solution does exist, is it unique? That is, is there exactly one solution to the problem or is there more than one solution?