Second Order Nonhomogeneous Differential Equations:  Section 3.4, 3.5

1. $z_1(x) = 2x^3 + x \ln x$, $z_2(x) = x \ln x - x^3$ are solutions of a second order, linear nonhomogeneous equation $L[y] = f(x)$. $y_1(x) = x^{-2}$ is a solution of the corresponding reduced equation $L[y] = 0$.

(a) Give a fundamental set of solutions of the reduced equation $L[y] = 0$. (Hint: The difference of two solutions of a nonhomogeneous equation is a solution of its reduced equation.)

(b) Give the general solution of the nonhomogeneous equation $L[y] = f(x)$.

Answer: (a) \{ $y_1(x) = x^{-2}$, $y_2(x) = x^3$ \}  (b) $y = C_1 x^{-2} + C_2 x^3 + x \ln x$

2. $z_1(x) = 2x^2 + \tan x$, $z_2(x) = x^2 - 2x + \tan x$, $z_3(x) = x^2 - 3x + \tan x$ are solutions of a second order, linear nonhomogeneous equation $L[y] = f(x)$.

(a) Give a fundamental set of solutions of the corresponding reduced equation $L[y] = 0$. (See the hint in # 1.)

(b) Give the general solution of the nonhomogeneous equation $L[y] = f(x)$.

Answer: (a) \{ $y_1(x) = x^2$, $y_2(x) = x$ \}  (b) $y = C_1 x^2 + C_2 x + \tan x$

3. Given the differential equation $y'' + p(x)y' + q(x)y = 4x$. The functions \{ $y_1 = x^2$, $y_2 = x^2 \ln x$ \} are solutions of the reduced equation.

(a) Use the Wronskan to show that \{ $y_1 = x^2$, $y_2 = x^2 \ln x$ \} is a fundamental set of solutions of the reduced equation.

(b) Find the general solution of the given equation.

Answer: (a) $W(x) = x^3$  (b) $y = C_1 x^2 + C_2 x^2 \ln x + 4x^3$.

4. Given the differential equation $y'' - \frac{5}{x} y' + \frac{8}{x^2} y = 2x^2$. The reduced equation has solutions of the form $y = x^r$. Find the general solution of the given equation

Answer: $y = C_1 x^2 + C_2 x^4 + x^4 \ln x$.

5. Find a particular solution of $y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 6x + 2$.

Answer: $z = 6x^3 \ln x - 2x^2 \ln x$.

6. Find the general solution of $y'' + 4y = 2 \tan 2x$.

Answer: $y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2} \cos 2x \ln |\sec 2x + \tan 2x|$.
7. Find the general solution of $y'' - 6y' + 9y = \frac{e^{3x}}{x}$.

**Answer:** $y = C_1 e^{3x} + C_2 x e^{3x} + x e^{3x} \ln x$.

8. Find the general solution of $y'' + 9y = 4 \cos 2x$.

**Answer:** $y = C_1 \cos 3x + C_2 \sin 3x + \frac{4}{9} \cos 2x$.

9. Find the general solution of $y'' + 4y = 2 \sin 2x$.

**Answer:** $y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2}x \cos 2x$.

10. Find the general solution of $y'' - 6y' + 8y = 2e^{4x} + 6$.

**Answer:** $y = C_1 e^{4x} + C_2 e^{2x} + x e^{4x} + \frac{3}{4}$.

11. A particular solution of the nonhomogeneous differential equation $y'' - 2y' - 15y = 2 \cos 3x + 5 e^{5x} + 2$ will have the form:

**Answer:** $z = A \cos 3x + B \sin 3x + Cx e^{5x} + D$.

12. A particular solution of the nonhomogeneous differential equation $y'' - 8y' + 16y = e^{2x} \sin 4x + 2 e^{4x} + 5x$ will have the form:

**Answer:** $A e^{2x} \cos 4x + B e^{2x} \sin 4x + C x^2 e^{4x} + D x + E$.

**Higher Order Linear Equations: Section 3.7**

1. The general solution of $y''' - 4y'' + y' + 6y = 0$ is: (Hint: $7e^{2x}$ is a root of the characteristic equation)

**Answer:** $y = C_1 e^{2x} + C_2 e^{3x} + C_3 e^{-x}$

2. The general solution of $y''' + y'' - 8y' - 12y = 0$ is: (Hint: $r = 3$ is a root of the characteristic equation)

**Answer:** $y = C_1 e^{3x} + C_2 e^{-2x} + C_3 x e^{-2x}$

3. The general solution of $y^{(4)} + 2y''' + 4y'' - 2y' - 5y = 0$ is: (Hint: $e^{-x} \cos 2x$ is a solution)

**Answer:** $y = C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x + C_3 e^x + C_4 e^{-x}$

4. The homogeneous equation with constant coefficients that has

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 \cos 2x + C_4 \sin 2x + C_5$$

as its general solution is:

**Answer:** $y^{(5)} + 4y^{(4)} + 8y''' + 16y'' + 16y' = 0$
5. The homogeneous equation with constant coefficients of least order that has
\[ y = 2e^{3x} + 3\sin 2x \]
as a solution is:

**Answer:** \[ y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 0 \]

6. A particular solution of \[ y''' - 2y'' - 3y' = 2e^{-x} + xe^{3x} + 2 \] will have the form:

**Answer:** \[ z = Axe^{-x} + (Bx^2 + Cx)e^{3x} + Dx \]

7. A particular solution of \[ y^{(4)} - 16y = 2e^{-2x} + 3e^{4x} + \cos 2x + 5 \] will have the form:

**Answer:** \[ z = Axe^{-2x} + Be^{4x} + Cx \cos 2x + Dx \sin 2x + E \]

8. The general solution of \[ y^{(4)} + 5y'' - 36y = -2x + 3x^2e^{2x} \] will have the form:

**Answer:** \[ y = C_1 \cos 3x + C_2 \sin 3x + C_3 e^{-2x} + Ax \cos x + Bx \sin 3x + (Cx^2 + Dx)e^{2x} \]

9. The general solution of \[ y''' + y'' + y' + y = 5 \sin x + 2e^x - e^{-x} + 4x \] will have the form:

**Answer:** \[ y = C_1 \cos x + C_2 \sin x + C_3 e^{-x} + Ax \cos x + Bx \sin x + Cx^{2} + Dx e^{-x} + Ex + F \]

**Laplace Transformations: Chapter 4**

1. Find the Laplace transform of \[ f(x) = 2e^{-3x} + \cos 2x + 5x. \]

**Answer:** \[ F(s) = \frac{2}{s+3} + \frac{s}{s^2 + 4} + \frac{5}{s^2}. \]

2. Find the Laplace transform of \[ f(x) = 3xe^{2x} + e^{x} \sin 3x. \]

**Answer:** \[ F(s) = \frac{3}{(s-2)^2} + \frac{3}{s^2 - 2s + 10}. \]

3. If \[ F(s) = \frac{2}{s^2} + \frac{s - 3}{s^2 + 4}, \] then \[ \mathcal{L}^{-1}[F(s)] \] is:

**Answer:** \[ f(x) = 2x + \cos 2x - \frac{3}{2} \sin 2x. \]

4. If \[ F(s) = \frac{4}{s^4 - 3s^3 + 2s^2}, \] then \[ \mathcal{L}^{-1}[F(s)] \] is:

**Answer:** \[ f(x) = 3 + 2x + e^{2x} - 4x \]

5. If \[ F(s) = \frac{3s^3 + 6s^2 + 36}{s^4 + 9s^2}, \] then \[ \mathcal{L}^{-1}[F(s)] \] is:

**Answer:** \[ f(x) = 4x + 3 \cos 3x + \frac{3}{3} \sin 3x. \]
6. If $F(s) = \frac{2s + 1}{(s + 3)(s^2 + 1)}$, then $\mathcal{L}^{-1}[F(s)]$ is:

Answer: $f(x) = \frac{1}{2} e^{-3x} + \frac{1}{2} \cos x + \frac{1}{7} \sin x$

7. If $F(s) = \frac{2s + 1}{s^3 - s^2 - 8s + 12}$, then $\mathcal{L}^{-1}[F(s)]$ is: (Hint: 2 is a root)

Answer: $f(x) = \frac{1}{5} e^{2x} + x e^{2x} - \frac{1}{9} e^{-3x}$

8. If $F(s) = \frac{3s + 1}{s^3 - 6s^2 + 13s - 20}$, then $\mathcal{L}^{-1}[F(s)]$ is: (Hint: 4 is a root)

Answer: $f(x) = e^{4x} - e^x \cos 2x$

9. Find the Laplace transform of the solution of the initial-value problem

$$y' + 2y = 3 \cos 2x; \quad y(0) = 3.$$ 

Answer: $Y(s) = \frac{9}{4(s + 2)} + \frac{3s + 6}{4(s^2 + 4)}$.

10. Find the Laplace transform of the solution of the initial-value problem

$$y'' - 5y' + 6y = 4 \sin 3x; \quad y(0) = 0, \quad y'(0) = 2.$$ 

Answer: $F(s) = \frac{12}{(s^2 + 9)(s^2 - 5s + 6)} + \frac{2}{s^2 - 5s + 6}$.

11. Find the Laplace transform of the solution of the initial-value problem

$$y'' + 25y = 2 e^{-3x}; \quad y(0) = 2, \quad y'(0) = 0.$$ 

Answer: $F(s) = \frac{2}{(s + 3)(s^2 + 25)} + \frac{2s}{s^2 + 25}$.

12. Use the Laplace transform method to find the solution of the initial-value problem

$$y' - 3y = 2 e^{2x}; \quad y(0) = 1.$$ 

Answer: $y = 3e^{3x} - 2e^{2x}$.

13. Use the Laplace transform method to find the solution of the initial-value problem

$$y' + 4y = 3 \cos 2x; \quad y(0) = 3.$$ 

Answer: $\frac{12}{5} e^{-4x} + \frac{3}{5} \cos 2x + \frac{3}{10} \sin 2x$. 


14. Use the Laplace transform method to find the solution of the initial-value problem
\[ y'' - 3y' + 2y = 2x + 1, \quad y(0) = 2, \quad y'(0) = -1. \]

**Answer:** \[ y = 2e^x - 2e^{2x} + x + 2. \]

15. Find the value(s) of \( \gamma \) such that the solution of the initial-value problem
\[ y'' - 4y = \sin x; \quad y(0) = \gamma, \quad y'(0) = 0 \]
is bounded on \([0, \infty)\).

**Answer:** \( \gamma = -\frac{1}{10}. \)

16. Find the value of \( \delta \) such that the solution of the initial-value problem
\[ y' - 3y = 2e^{-2x}; \quad y(0) = \delta \]
has limit 0 as \( x \to \infty \).

**Answer:** \( \delta = -\frac{2}{5}. \)

17. If
\[ f(x) = \begin{cases} x^2 + 1 & 0 \leq x < 3 \\ 2x & x \geq 3 \end{cases} \]
then \( \mathcal{L}[f(x)] = \)

**Answer:** \[ F(s) = \frac{2}{s^3} + \frac{1}{s}e^{-3s} + \frac{2}{s^3} - 4e^{-3s} + \frac{1}{s^2} - 4e^{-3s}\frac{1}{s}. \]

18. If
\[ f(x) = \begin{cases} \sin x & 0 \leq x < \pi/2 \\ \cos 2x & x \geq \pi/2 \end{cases} \]
then \( \mathcal{L}[f(x)] = \)

**Answer:** \[ F(s) = \frac{1}{s^2 + 1} - e^{\pi s/2}s + \frac{s}{s^2 + 1} - e^{\pi s/2}\frac{s}{s^2 + 4}. \]

19. If
\[ f(x) = \begin{cases} -2 & 0 \leq x < 2 \\ x & 2 \leq x < 5 \\ 3 & x \geq 5 \end{cases} \]
then \( \mathcal{L}[f(x)] = \)

**Answer:** \[ F(s) = -\frac{2}{s} - e^{-2s}\frac{1}{s^2} + 4e^{-2s}\frac{1}{s} - e^{-5s}\frac{1}{s^2} - 2e^{-5s}\frac{1}{s}. \]
20. If 
\[ f(x) = \begin{cases} 
  x^2 + 1 & 0 \leq x < 2 \\
  4e^{3x} & x \geq 2
\end{cases} \]
then \( \mathcal{L}[f(x)] = \)

**Answer:** 
\[ F(s) = \frac{2}{s^3} + \frac{1}{s} e^{-2s} \frac{2}{s^3} - 4 e^{-2s} \frac{1}{s^2} - 5 - 2s \frac{1}{s} + 4e^{6} e^{-2s} \frac{1}{s - 3}. \]

21. If \( F(s) = \frac{2}{s^2} + \frac{6}{s^3} + 3e^{-2s} \frac{1}{s^2} + 4 e^{-2s} \frac{1}{s^2 - 2s - 8} \), then \( \mathcal{L}^{-1}[F(s)] \) is:

**Answer:** 
\[ f(x) = \begin{cases} 
  2x + 3x^2 & 0 \leq x < 2 \\
  3x^2 + 5x - 6 + \frac{2}{3} e^{4x-8} - \frac{2}{3} e^{-2x+4} & x \geq 2
\end{cases} \]

22. If \( F(s) = \frac{s + 4e^{-3s}}{s^3 - 2s^2} \), then \( \mathcal{L}^{-1}[F(s)] \) is:

**Answer:** 
\[ f(x) = \begin{cases} 
  \frac{1}{2} e^{2x} - \frac{1}{2} & 0 \leq x < 3 \\
  \frac{1}{2} e^{2x} + e^{2(x-3)} - 2x + \frac{9}{2}, & x \geq 3
\end{cases} \]

23. If \( F(s) = \frac{3s + 1}{s^2 - s - 6} + \frac{(2s - 4)e^{-4s}}{s^2 - 2s + 10} \), then \( \mathcal{L}^{-1}[F(s)] \) is:

**Answer:** 
\[ f(x) = \begin{cases} 
  2 e^{3x} + e^{-2x}, & 0 \leq x < 4 \\
  2 e^{3x} + e^{-2x} + 2 e^{x-4} \cos 3(x - 4) - \frac{2}{3} e^{x-4} \sin 3(x - 4), & x \geq 4
\end{cases} \]