MATH 3321
Sample Questions for Exam 3

1. Find the solution set of the system of linear equations
   \begin{align*}
   4x - y + 2z &= 3 \\
   -4x + y - 3z &= -10 \\
   8x - 2y + 9z &= -1 
   \end{align*}

   \textbf{Answer:} No solution

2. Find the solution set of the system of linear equations
   \begin{align*}
   2x - 5y - 3z &= 7 \\
   -4x + 10y + 2z &= 6 \\
   6x - 15y - z &= -19 
   \end{align*}

   \textbf{Answer:} \( x = \frac{5}{2}a - 4, \ y = a, \ z = -5, \ a \) any real number

3. Find the solution set of the system of linear equations
   \begin{align*}
   5x - 3y + 2z &= 13 \\
   2x - y - 3z &= 1 \\
   4x - 2y + 4z &= 12 
   \end{align*}

   \textbf{Answer:} \( x = 1, \ y = -2, \ z = 1 \)

4. Find the solution set of the system of linear equations
   \begin{align*}
   x_1 + 2x_2 - x_3 - x_4 &= 0 \\
   x_1 + 2x_2 + x_4 &= 4 \\
   -x_1 - 2x_2 + 2x_3 + 4x_4 &= 5 
   \end{align*}

   \textbf{Answer:} \( x_1 = 3 - 2a, \ x_2 = a, \ x_3 = 2, \ x_4 = 1, \ a \) any real number

5. Given the system of equations
   \begin{align*}
   x_1 - 2x_2 + x_3 - x_4 &= -2 \\
   -2x_1 + 5x_2 - x_3 + 4x_4 &= 1 \\
   3x_1 - 7x_2 + 2x_3 - 5x_4 &= 9 \\
   2x_2 + x_3 + 5x_4 &= -2 
   \end{align*}

   Find the rank of the matrix of coefficients and the rank of the augmented matrix. Does the system have a unique solution, infinitely many solutions, or no solutions.

   \textbf{Answer:} 3, 4; no solutions
6. Find the row echelon form of \[
\begin{pmatrix}
1 & 2 & 3 & 2 \\
-1 & -2 & -2 & 1 \\
2 & 4 & 8 & 12
\end{pmatrix}
\]

Answer: \[
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

7. Find the reduced row echelon form of \[
\begin{pmatrix}
1 & 2 & -3 & 4 & 2 \\
2 & 5 & -2 & 1 & 1 \\
5 & 12 & -7 & 6 & 7
\end{pmatrix}
\]

Answer: \[
\begin{pmatrix}
1 & 0 & -11 & 18 & 0 \\
0 & 1 & 4 & -7 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

8. Find the solution set of the system of linear equations

\[
\begin{align*}
x_1 + 2x_2 - 3x_3 - 4x_4 &= 2 \\
2x_1 + 4x_2 - 5x_3 - 7x_4 &= 7 \\
-3x_1 - 6x_2 + 11x_3 + 14x_4 &= 0
\end{align*}
\]

Answer: \[x_1 = 11 - 2a + b, \ x_2 = a, \ x_3 = 3 - b, \ x_4 = b, \ a, b \text{ any real numbers} \]

9. Find the solution set of the homogeneous system of linear equations

\[
\begin{align*}
3x_1 + x_2 - 5x_3 - x_4 &= 0 \\
2x_1 + x_2 - 3x_3 - 2x_4 &= 0 \\
x_1 + x_2 - x_3 - 3x_4 &= 0
\end{align*}
\]

Answer: \[x_1 = 2a - b, \ x_2 = -a + 4b, \ x_3 = a, \ x_4 = b, \ a, b \text{ any real numbers} \]

10. For what values of \( k \), does the system of equations

\[
\begin{align*}
x - 2y &= 1 \\
x - y + kz &= -2 \\
kx + 4z &= 6
\end{align*}
\]

have (a) a unique solution? (b) infinitely many solutions? (c) no solutions?

Answer: (a) \( k \neq \pm 2 \) (b) \( k = -2 \) (c) \( k = 2 \)

11. For what values of \( k \), does the system of equations

\[
\begin{align*}
x + 2y + 3z &= 4 \\
y + 5z &= 9 \\
2x + 3y + (k^2 - 8)z &= k + 2
\end{align*}
\]

have (a) a unique solution? (b) infinitely many solutions? (c) no solutions?

Answer: (a) \( k \neq \pm 3 \) (b) \( k = -3 \) (c) \( k = 3 \)
12. Let \( A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{pmatrix} \), \( B = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix} \), \( C = \begin{pmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \).

Perform the indicated operations, if possible:  
(a) \( AC \)  
(b) \( AB \)  
(c) \( B + AC \)  
(d) \( CBA \)

**Answer:**  
\( AC = \begin{pmatrix} 9 & 3 \\ -2 & -8 \end{pmatrix} \), \( AB \) no, \( B + AC = \begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix} \), \( CBA = \begin{pmatrix} -26 & -15 & -25 \\ -4 & -18 & 4 \\ 2 & 43 & -19 \end{pmatrix} \)

13. Find the inverse of \( A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} \).

**Answer:** \( A^{-1} = \begin{pmatrix} -1 & 3/2 \\ 2 & -5/2 \end{pmatrix} \)

14. Determine whether or not \( A = \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix} \) has an inverse.

15. Find the inverse of \( A = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix} \).

**Answer:** \( A^{-1} = \begin{pmatrix} -16 & -11 & 3 \\ 7/2 & 5/2 & -1/2 \\ -5/2 & -3/2 & 1/2 \end{pmatrix} \)

16. Show that the matrix of coefficients is non-singular and use Cramer’s rule to find the values of \( y \) and \( z \) in the solution set,

\[
\begin{align*}
2x_1 - x_2 + x_3 &= 1 \\
2x_2 - x_3 &= 1 \\
2x_1 + 3x_2 &= 1
\end{align*}
\]

**Answer:** \( y = -3, \ z = -7 \)

17. Find the values of \( \lambda \) for which the homogeneous system

\[
\begin{align*}
(2 - \lambda)x - 3y &= 0 \\
4x + (2 - \lambda)y &= 0
\end{align*}
\]

has only the trivial solution.

**Answer:** \( \lambda \neq 5, -2 \)

18. Find the values of \( \lambda \) for which the system

\[
\begin{align*}
\lambda x + 2y - 4z &= 0 \\
-x + y + \lambda z &= 0 \\
-y + 5z &= 0
\end{align*}
\]
has nontrivial solutions.

**Answer:** $\lambda = -2, \lambda = -3$

19. Given the set of vectors

$$\{ \mathbf{u} = (1, -2, 1), \mathbf{v} = (2, 1, -1), \mathbf{w} = (7, -4, 1) \}$$

Is the set linearly dependent or linearly independent? If it is linearly dependent, express one of the vectors as a linear combination of the other two.

**Answer:** Dependent, $\mathbf{u} = -\frac{2}{3} \mathbf{v} + \frac{1}{3} \mathbf{w}$ or $\mathbf{v} = -\frac{3}{2} \mathbf{u} + \frac{1}{2} \mathbf{w}$ or $\mathbf{w} = 3\mathbf{u} + 2\mathbf{v}$

20. Given the set of vectors

$$\{ \mathbf{v}_1 = (2, 0, -1), \mathbf{v}_2 = (-3, 1, 2), \mathbf{v}_3 = (8, -2, -5), \mathbf{v}_4 = (-9, 1, 5) \}$$

Is the set linearly dependent or linearly independent? If it is linearly dependent, how many independent vectors are there in the set?

**Answer:** Dependent, 2

21. For what values of $a$ are the vectors

$$\mathbf{v}_1 = (a, 1, -1), \mathbf{v}_2 = (-1, 2a, 3), \mathbf{v}_3 = (-2, a, 2), \mathbf{v}_4 = (3a, -2, a)$$

linearly dependent?

**Answer:** All real numbers. (Four vectors in $\mathbb{R}^3$.)

22. For what values of $a$ are the vectors

$$\mathbf{u} = (a, 1, -1), \mathbf{v} = (-1, 2a, 3), \mathbf{w} = (-2, a, 2)$$

linearly dependent?

**Answer:** $a = 4$, $-1$

23. The eigenvalues of a $3 \times 3$ matrix $A$ are $\lambda_1 = 4, \lambda_2 = -2, \lambda_3 = 2$. What is the characteristic polynomial of $A$?

**Answer:** $p(\lambda) = (\lambda - 4)(\lambda + 2)(\lambda - 2)$ (which is $\lambda^3 - 4\lambda^2 - 4\lambda + 16$ if you multiply it out.)

24. The eigenvalues of a $3 \times 3$ matrix $A$ are $\lambda_1 = -3, \lambda_2 = \lambda_3 = 2$. What is the characteristic polynomial of $A$?

**Answer:** $p(\lambda) = (\lambda + 3)(\lambda - 2)^2$ (which is $\lambda^3 - \lambda^2 - 8\lambda + 12$ if you multiply it out.)

25. Find the eigenvalues and the number of independent eigenvectors of $A = \begin{pmatrix} 3 & 5 & -2 \\ 0 & 2 & 2 \\ 0 & -2 & 6 \end{pmatrix}$.

**Answer:** $\lambda_1 = 3, \lambda_2 = \lambda_3 = 4$; two independent eigenvectors.
26. The eigenvalues and eigenvectors of \[
\begin{pmatrix}
6 & -2 & 0 \\
4 & 0 & 0 \\
4 & 1 & -1
\end{pmatrix}
\] are:

Answer: 4, \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]; 2, \[
\begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix}
\]; -1, \[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

27. The eigenvalues and eigenvectors of \[
\begin{pmatrix}
3 & -2 & 1 \\
2 & -2 & 2 \\
2 & -3 & 3
\end{pmatrix}
\] are: (Hint: 1 is an eigenvalue.)

Answer: 2, \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]; 1, 1, \[
\begin{pmatrix}
1 \\
2 \\
2
\end{pmatrix}
\]

28. The eigenvalues and eigenvectors of \[
\begin{pmatrix}
4 & 2 & 2 \\
2 & 4 & 2 \\
2 & 2 & 4
\end{pmatrix}
\] are: (Hint: 2 is an eigenvalue.)

Answer: 8, \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]; 2, \[
\begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}
\]; 2, \[
\begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}
\]

29. The eigenvalues and eigenvectors of \[
\begin{pmatrix}
1 & -2 \\
5 & 3
\end{pmatrix}
\] are:

Answer: \(2 + 3i\), \[
\begin{pmatrix}
-1 \\
5
\end{pmatrix}
\] + \(i\) \[
\begin{pmatrix}
3 \\
0
\end{pmatrix}
\]; \(2 - 3i\), \[
\begin{pmatrix}
-1 \\
5
\end{pmatrix}
\] - \(i\) \[
\begin{pmatrix}
3 \\
0
\end{pmatrix}
\]

30. Find the general solution of \(x' = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} x\).

Answer: \(x = C_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\).

31. Find the solution of the initial-value problem \(x' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} x; x(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\). HINT: 

\(2\) is a root of the characteristic polynomial.

Answer: General solution: \(x = C_1 e^{3t} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{t} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\);

Solution of the initial-value problem: \(x = 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - e^{t} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\)
32. Find the general solution of $x' = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix} x$.

**Answer:** $x(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left[ e^{3t/2} \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} + t e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$

33. Find a fundamental set of solutions of $x' = \begin{pmatrix} 1 & 4 \\ -2 & 5 \end{pmatrix} x$.

**Answer:** \( \left\{ e^{3t} \left[ \cos 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right], e^{3t} \left[ \cos 2t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \right\} \)

34. Find the general solution of $x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x$. HINT: 2 is a root of the characteristic polynomial.

**Answer:** $x = C_1 e^{-t} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 \left[ e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right]$

35. Find a fundamental set of solutions of $x' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} x$. HINT: 10 is a root of the characteristic polynomial.

**Answer:** \( \left\{ e^{10t} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, e^t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, e^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \)