POLYNOMIALS

(Polynomials with Real Coefficients)

Definition 1: A *real polynomial* is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_{n-1}, a_n$ are real numbers with $a_n \neq 0$. The nonnegative integer n is called the *degree* of P. The numbers $a_0, a_1, \ldots, a_{n-1}, a_n$ are called the *coefficients* of P; a_n is called the *leading coefficient*.

Examples:

- Polynomials of degree 0: The non-zero constants $P(x) \equiv a$. Note: $P(x) \equiv 0$ (the zero polynomial) is a polynomial but no degree is assigned to it.
- Polynomials of degree 1: Linear polynomials $P(x) = ax + b$. The graph of a linear polynomial is a straight line.
- Polynomials of degree 2: Quadratic polynomials $P(x) = ax^2 + bx + c$. The graph of a quadratic polynomial is a parabola which opens up if $a > 0$, down if $a < 0$.
- Polynomials of degree 3: Cubic polynomials $P(x) = ax^3 + bx^2 + cx + d$.
- Polynomials of degree 4: Quartic polynomials $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.
- Polynomials of degree 5: Quintic polynomials

  $$P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

- And so on.

Definition 2: Let P be a polynomial of degree $n \geq 1$. A number r such that $P(r) = 0$ is called a *root* or *zero* of P.

Examples:

- $r = -\dfrac{b}{a}$ is a root of the linear polynomial $P(x) = ax + b$.
- $r_1,\, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (the quadratic formula) are the roots of the quadratic polynomial

  $$P(x) = ax^2 + bx + c.$$

- There is a cubic formula and a quartic formula. There are no formulas for the roots of a polynomial of degree $n \geq 5$.

THEOREM 1. (**Factor Theorem**) A number r is a root of the polynomial P (of degree n) if and only if $(x-r)$ is a factor of P. That is, r is a root of P if and only if

$$P(x) = (x-r)Q(x)$$

where Q is a polynomial of degree $n-1$.

Definition 3: Suppose P is a polynomial of degree n. A number r is a root of P of *multiplicity* k if and only if $(x-r)^k$ is a factor of P and $(x-r)^{k+1}$ is not a factor of P. That is, r is a root of P of multiplicity k if and only if

$$P(x) = (x-r)^k Q(x)$$

where Q is a polynomial of degree $m = n-k$ and r is not a root of Q.

NOTE: A polynomial with real coefficients may have roots that are complex numbers. For example, the roots of the quadratic polynomial

$$P(x) = x^2 - 2x + 5$$

are $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$.

THEOREM 2. **(Complex Root Theorem)** If $r_1 = \alpha + \beta i$ is a root of the polynomial P, then $r_2 = \alpha - \beta i$ (the conjugate of r_1) is also a root of P; the complex roots of P occur in conjugate pairs.

COROLLARY: A polynomial of odd degree must have at least one real root.

THEOREM 3. A polynomial of degree $n \geq 1$ has exactly n roots, counting multiplicities. The roots may be either real numbers or complex numbers, with any complex roots occurring in conjugate pairs.