Final Exam Review Questions

1. \(y^2 = Cx^3 - 3x\) is the general solution of a differential equation. Find the equation.
   \[
   y' = \frac{3y^2 + 6x}{2xy}
   \]

2. \(y = C_1x^3 + C_2 - 2x\) is the general solution of a differential equation. Find the equation.
   \[
   xy'' - 2y' = 4
   \]

3. Identify and find the general solution of each of the following first order differential equations.
   (a) \(xy' = 5x^3y^{1/2} - 4y\)
   \[
   \text{Answer: Bernoulli, } y^{1/2} = \frac{1}{2} x^3 + \frac{C}{x^2}
   \]
   (b) \(x y' + 3y = \cos \frac{2x}{x^2}\)
   \[
   \text{Answer: linear, } y = \frac{\sin 2x}{2x^3} + \frac{C}{x^3}
   \]
   (c) \(x^2 \frac{dy}{dx} = x^2 + xy + y^2\)
   \[
   \text{Answer: homogeneous, } y = x \tan(\ln x + C)
   \]
   (d) \(2y' = \frac{y^2 + 3}{4y + xy}\)
   \[
   \text{Answer: separable, } y^2 = C(4 + x) - 3
   \]
   (e) \(x^2y' = 4x^3y^3 + xy\)
   \[
   \text{Answer: Bernoulli, } y^2 = \frac{x^2}{C - 2x^4}
   \]
   (f) \(x^3y' = x^2y + 2x^3e^{y/x}\)
   \[
   \text{Answer: homogeneous, } y = x \ln \left(\frac{1}{C - \ln x^2}\right)
   \]

4. Given the one-parameter family \(y^3 = Cx^2 + 4\).
   (a) Find the differential equation for the family.
   (b) Find the differential equation for the family of orthogonal trajectories.
   (c) Find the family of orthogonal trajectories.
   \[
   \text{Answer: (a) } y' = \frac{2y^3 - 8}{3xy^2} \quad \text{(b) } y' = \frac{-3xy^2}{2y^3 - 8} \quad \text{(c) } 3x^2 + 2y^2 + \frac{16}{y} = C
   \]

5. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 100 grams of the material was present initially and after 3 hours the sample lost 30\% of its mass, find:
   (a) An expression for the mass \(A(t)\) of the material remaining at any time \(t\).
(b) The mass of the material after 8 hours.
(c) The half-life of the material.

Answer: (a) \( A(t) = 100 \left( \frac{7}{10} \right)^{t/3} \), (b) \( A(8) = 100 \left( \frac{7}{10} \right)^{8/3} \approx 30.63 \), (c) \( T = \frac{-3 \ln 2}{\ln(7/10)} \)

6. Scientists observed that a colony of penguins on a remote Antarctic island obeys the population growth law. There were 1000 penguins in the initial population and there were 3000 penguins 4 years later.

(a) Give an expression for the number \( P(t) \) of penguins at any time \( t \).
(b) How many penguins will there be after 6 years (to the nearest penguin)?
(c) How long will it take for the number of penguins to quadruple?

Answer: (a) \( P(t) = 1000(3)^{t/4} \), (b) \( P(6) = 1000(3)^{3/2} \approx 5,196 \), (c) \( t = \frac{4 \ln 4}{\ln 3} \) years

7. A disease is infecting a herd of 1000 cows. Let \( P(t) \) be the number of sick cows \( t \) days after the outbreak. Suppose that 50 cows had the disease initially, and suppose that the disease is spreading at a rate proportional to the product of the time elapsed and the number of cows who do not have the disease.

(a) Give the mathematical model (initial-value problem) for \( P \).
(b) Find the solution of the initial-value problem in (a).

Answer: (a) \( \frac{dP}{dt} = kt(1000 - P), P(0) = 50 \), (b) \( P(t) = 1000 - 950e^{-kt^2/2} \)

8. Determine a fundamental set of solutions of \( y'' - 2y' - 15y = 0 \).

Answer: \( \{y_1 = e^{5x}, y_2 = e^{-3x}\} \)

9. Find the general solution of \( y'' + 6y' + 9y = 0 \).

Answer: \( y = C_1 e^{-3x} + C_2 xe^{-3x} \)

10. Find the general solution of \( y'' + 4y' + 20y = 0 \)

Answer: \( y = C_1 e^{-2x} \cos 4x + C_2 e^{-2x} \sin 4x \)

11. Find the solution of the initial-value problem \( y'' - 7y' + 12y = 0, y(0) = 3, y'(0) = 0 \).

Answer: \( y = 12e^{3x} - 9e^{4x} \).

12. The function \( y = -2 e^{-3x} \sin 2x \) is a solution of a second order, linear, homogeneous differential equation with constant coefficients. What is the equation?

Answer: \( y'' + 6y' + 13y = 0 \)
13. The function

\[ y = 4xe^{-4x} \]

is a solution of a second order, linear, homogeneous differential equation with constant coefficients. What is the equation?

**Answer:** \[ y'' + 8y' + 16y = 0 \]

14. Find a particular solution of \( y'' - 6y' + 8y = 4e^{4x} \).

**Answer:** \( z = 2xe^{4x} \)

15. Give the form of a particular solution of the nonhomogeneous differential equation

\[ y'' - 8y' + 16y = 2e^{4x} + 3\cos 4x - 2x + 1. \]

**Answer:** \( z = Ax^2e^{4x} + B\cos 4x + C\sin 4x + Dx + E \)

16. Given the differential equation

\[ y'' - 4y' + 4y = \frac{e^{2x}}{x} \]

(a) Give the general solution of the reduced equation.

(b) Find a particular solution of the nonhomogeneous equation.

**Answer:** (a) \( y = C_1e^{2x} + C_2xe^{2x} \) (b) \( z = xe^{2x}\ln x \)

17. Find the general solution of \( y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 4x \). HINT: The reduced equation has solutions of the form \( y = x^r \).

**Answer:** \( y = c_1x^2 + c_2x^3 + 4x^3\ln x \)

18. Find a particular solution of \( y'' + 4y = 2\tan 2x \).

**Answer:** \( z = -\frac{1}{2}\cos 2x\ln|\sec 2x + \tan 2x| \)

19. The general solution of

\[ y^{(4)} - 6y''' + 17y'' - 28y' + 20y = 0 \]

is: (HINT: 2 is a root of the characteristic polynomial)

**Answer:** \( y = C_1e^{2x} + C_2xe^{2x} + C_3e^x\cos 2x + C_4e^x\sin 2x \)

20. Find the linear homogeneous equation of least order that has \( y = 2xe^{-3x} + 4\cos 2x + 9x \) as a solution.

**Answer:** \( y^{(6)} + 6y^{(5)} + 13y^{(4)} + 24y''' + 36y'' = 0 \)

21. Find the Laplace transform of the solution of the initial-value problem

\[ y'' - 3y' - 6y = 2e^{2x} + 4; \ y(0) = 5, \ y'(0) = -2 \]

**Answer:** \( Y(s) = \frac{2}{(s - 2)(s^2 - 3s - 6)} + \frac{4}{s(s^2 - 3s - 6)} + \frac{5s - 17}{s^2 - 3s - 6} \)
22. Find \( f(x) = \mathcal{L}^{-1}[F(s)] \) if \( F(s) = \frac{3}{s^2} + \frac{4s + 3}{s^2 + 4} \).

**Answer:** \( f(x) = 3x + 4\cos 2x + \frac{1}{2}\sin 2x \)

23. Find \( f(x) = \mathcal{L}^{-1}[F(s)] \) if \( F(s) = \frac{s^2 - 3s - 1}{(s - 2)^2(s + 4)} \)

**Answer:** \( f(x) = \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + \frac{3}{4}e^{-4x} \)

24. Find \( L[f(x)] \) if

\[
f(x) = \begin{cases} 
  x^2 + 2x & 0 \leq x < 4 \\
  x & x \geq 4 
\end{cases}
\]

**Answer:** \( F(s) = \frac{2}{s^3} + \frac{2}{s^2} - e^{-4s}\frac{2}{s^3} - 9e^{-4s}\frac{1}{s} - 20e^{-4s}\frac{1}{s} \)

25. \( F(s) = \frac{5}{s^3} - \frac{3}{s} - 2e^{-3s}\frac{1}{s^3} + 3e^{-3s}\frac{1}{s} + 2e^{-3s}\frac{s + 1}{s^2 + \pi^2} \). Find \( \mathcal{L}^{-1}[F(s)] = f(x) \).

**Answer:** \( f(x) = \begin{cases} 
  5x - 3, & 0 \leq x < 3 \\
  3x + 6 - 2\cos \pi x - \frac{2}{\pi}\sin \pi x, & x \geq 3
\end{cases} \)

26. Given the initial-value problem

\[ y' - 4y = 2e^{-2x}, \quad y(0) = 3. \]

(a) Find the Laplace transform of the solution.

(b) Find the solution by finding the inverse Laplace transform of your answer to (a).

**Answer:** (a) \( Y(s) = \frac{2}{(s + 2)(s - 4)} + \frac{3}{s - 4} \)

(b) \( y = \frac{10}{3}e^{4x} - \frac{4}{3}e^{-2x} \)

27. Given the system of equations

\[
\begin{align*}
x + 2y - z &= 1 \\
2x + 5y - 4z &= 3 \\
-2x - 2y - 2z &= 0
\end{align*}
\]

(a) Write the augmented matrix for the system.

(b) Reduce the augmented matrix to row-echelon form.

(c) Give the solution set of the system.

**Answer:** (a) \[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
2 & 5 & -4 & 3 \\
-2 & -2 & -2 & 0
\end{pmatrix}
\] (b) \[
\begin{pmatrix}
1 & 2 & -1 & 1 \\
0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(c) \( x = -1 - 3a, \ y = 1 + 2a, \ z = a \) a any real number
28. Determine the values of \( k \) so that the system of equations

\[
\begin{align*}
  x + y - z &= 1 \\
  2x + 3y + kz &= 3 \\
  x + ky + 3z &= 2
\end{align*}
\]

has: (a) a unique solution, (b) no solutions, (c) infinitely many solutions

\textbf{Answer:}  \( (a) \ k \neq -3, 2 \quad (b) \ k = -3 \quad (c) \ k = 2 \)

29. Find the values of \( \lambda \), (if any) such that \( A = \begin{pmatrix} \lambda & 0 & 3 \\ 0 & 1 & \lambda \\ -2 & -1 & -5 \end{pmatrix} \) is nonsingular.

\textbf{Answer:}  \( \lambda \neq 2, 3 \)

30. The matrix \( A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -3 \\ -2 & -4 & 1 \end{pmatrix} \) is nonsingular. Find \( A^{-1} \).

\textbf{Answer:}  \( A^{-1} = \begin{pmatrix} -7 & -2 & -6 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \)

31. The system of equations

\[
\begin{align*}
  2x - y + 3z &= 4 \\
  y + 2z &= -2 \\
  x + z &= 1
\end{align*}
\]

has a unique solution. Find \( y \).

\textbf{Answer:}  \( y = -2 \)

32. Determine whether the vectors

\[
\begin{align*}
  \mathbf{v}_1 &= (1, -3, 2), \quad \mathbf{v}_2 = (0, -2, -2), \\
  \mathbf{v}_3 &= (1, -5, 0), \quad \mathbf{v}_4 = (0, 4, 4)
\end{align*}
\]

are linearly dependent or linearly independent. If they are linearly dependent, find the maximal number of independent vectors.

\textbf{Answer:}  The vectors are linearly dependent. The maximum number of independent vectors is 2.
33. For what values of $a$ are the vectors 

\[ v_1 = (1, 3, a), \quad v_2 = (-a, -2, -2), \quad \]

\[ v_3 = (1, -1, 0) \]

linearly dependent?

**Answer:** $a = -4, 2$

34. Find the eigenvalues and eigenvectors of \( \begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix} \). Hint: 2 is an eigenvalue.

**Answer:** \( \lambda_1 = 2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \); \( \lambda_2 = -1, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \); \( \lambda_3 = 1, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \)

35. Find the eigenvalues and eigenvectors of \( \begin{pmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{pmatrix} \). Hint: 6 is an eigenvalue.

**Answer:** \( \lambda_1 = 6, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \); \( \lambda_2 = -2, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \); \( \lambda_3 = -2, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \)

36. Find the eigenvalues and eigenvectors of \( \begin{pmatrix} -2 & 1 & -1 \\ 3 & -3 & 4 \\ 3 & -1 & 2 \end{pmatrix} \). Hint: 1 is an eigenvalue.

**Answer:** \( \lambda_1 = 1, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \); \( \lambda_2 = \lambda_3 = -2, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \)

37. Find the solution of the initial-value problem \( x' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \). HINT: 1 is an eigenvalue.

**Answer:** \( x(t) = 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - e^t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \)

38. Find a fundamental set of solutions of \( x' = \begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix} x. \)

**Answer:** \( \left\{ e^{3t} \begin{pmatrix} \cos 2t & 1 \\ -\sin 2t & 0 \end{pmatrix}, e^{3t} \begin{pmatrix} \cos 2t & 0 \\ \sin 2t & 1 \end{pmatrix} \right\} \)
39. The system \( x' = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix} x \) is equivalent to a second order linear equation. (a) What is the equation? (b) Find the general solution of the system.

**Answer:** (a) \( y'' - 6y' + 8y = 0 \) (b) \( x(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \)

40. Find the general solution of \( x' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} x \).

**Answer:** \( x(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left[ e^{3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + te^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \)

41. Find the general solution of \( x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x \). HINT: 2 is a root of the characteristic polynomial.

**Answer:** \( x(t) = C_1 e^{-t} \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + te^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \)

42. Find a fundamental set of solutions of \( x' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} x \). HINT: 10 is a root of the characteristic polynomial.

**Answer:** \[ \left\{ e^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \ e^t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \ e^{10t} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\} \]