Final Exam Review Questions

1. $y^2 = Cx^3 - 3x$ is the general solution of a differential equation. Find the equation.

   Answer: $y' = \dfrac{3y^2 + 6x}{2xy}$

2. $y = C_1x^3 + C_2 - 2x$ is the general solution of a differential equation. Find the equation.

   Answer: $xy'' - 2y' = 4$

3. Identify and find the general solution of each of the following first order differential equations.

   (a) $xy' = 5x^3y^{1/2} - 4y$

   Answer: Bernoulli, $y^{1/2} = \dfrac{1}{2}x^3 + \dfrac{C}{x^2}$

   (b) $x\,y' + 3\,y = \dfrac{\cos\,2x}{x^2}$.

   Answer: linear, $y = \dfrac{\sin 2x}{2x^3} + \dfrac{C}{x^3}$

   (c) $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$

   Answer: homogeneous, $y = x\tan(\ln x + C)$

   (d) $2y' = \dfrac{y^2 + 3}{4y + xy}$.

   Answer: separable, $y^2 = C(4 + x) - 3$

   (e) $x^2y' = 4x^3y^3 + xy$

   Answer: Bernoulli, $y^2 = \dfrac{x^2}{C - 2x^4}$

   (f) $x^3y' = x^2y + 2x^3e^{y/x}$

   Answer: homogeneous, $y = x\ln\left(\dfrac{1}{C - \ln x^2}\right)$

4. Given the one-parameter family $y^3 = Cx^2 + 4$.

   (a) Find the differential equation for the family.

   (b) Find the differential equation for the family of orthogonal trajectories.

   (c) Find the family of orthogonal trajectories.

   Answer: (a) $y' = \dfrac{2y^3 - 8}{3xy^2}$ (b) $y' = \dfrac{-3xy^2}{2y^3 - 8}$ (c) $3x^2 + 2y^2 + \dfrac{16}{y} = C$

5. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 100 grams of the material was present initially and after 3 hours the sample lost 30% of its mass, find:

   (a) An expression for the mass $A(t)$ of the material remaining at any time t.

(b) The mass of the material after 8 hours.

(c) The half-life of the material.

Answer: (a) $A(t) = 100\left(\frac{7}{10}\right)^{t/3}$, (b) $A(8) = 100\left(\frac{7}{10}\right)^{8/3} \approx 30.63$ (c) $T = \frac{-3\ln 2}{\ln(7/10)}$

6. Scientists observed that a colony of penguins on a remote Antarctic island obeys the population growth law. There were 1000 penguins in the initial population and there were 3000 penguins 4 years later.

   (a) Give an expression for the number $P(t)$ of penguins at any time t.

   (b) How many penguins will there be after 6 years (to the nearest penguin)?

   (c) How long will it take for the number of penguins to quadruple?

   Answer: (a) $P(t) = 1000(3)^{t/4}$ (b) $P(6) = 1000(3)^{3/2} \approx 5,196$ (c) $t = \frac{4\ln 4}{\ln 3}$ years

7. A disease is infecting a herd of 1000 cows. Let $P(t)$ be the number of sick cows t days after the outbreak. Suppose that 50 cows had the disease initially, and suppose that the disease is spreading at a rate proportional to the product of the time elapsed and the number of cows who do not have the disease.

   (a) Give the mathematical model (initial-value problem) for P.

   (b) Find the solution of the initial-value problem in (a).

   Answer: (a) $\frac{dP}{dt} = kt(1000 - P), P(0) = 50,$ (b) $P(t) = 1000 - 950e^{-kt^2/2}$

8. Determine a fundamental set of solutions of $y'' - 2y' - 15y = 0$.

   Answer: $\{y_1 = e^{5x},\ y_2 = e^{-3x}\}$

9. Find the general solution of $y'' + 6y' + 9y = 0$.

   Answer: $y = C_1 e^{-3x} + C_2 x e^{-3x}$

10. Find the general solution of $y'' + 4y' + 20y = 0$

    Answer: $y = C_1 e^{-2x}\cos 4x + C_2 e^{-2x}\sin 4x$

11. Find the solution of the initial-value problem $y'' - 7y' + 12y = 0,\ y(0) = 3,\ y'(0) = 0$.

    Answer: $y = 12e^{3x} - 9e^{4x}$.

12. The function $y = -2\,e^{-3x}\sin 2x$ is a solution of a second order, linear, homogeneous differential equation with constant coefficients. What is the equation?

    Answer: $y'' + 6y' + 13y = 0$

13. The function

$$y = 4\,xe^{-4x}$$

is a solution of a second order, linear, homogeneous differential equation with constant coefficients. What is the equation?

Answer: $y'' + 8y' + 16y = 0$

14. Find a particular solution of $y'' - 6y' + 8y = 4e^{4x}$.

Answer: $z = 2xe^{4x}$

15. Give the form of a particular solution of the nonhomogeneous differential equation

$$y'' - 8\,y' + 16\,y = 2\,e^{4x} + 3\,\cos\,4x - 2x + 1.$$

Answer: $z = Ax^2e^{4x} + B\cos 4x + C\sin 4x + Dx + E$

16. Given the differential equation

$$y'' - 4\,y' + 4\,y = \frac{e^{2x}}{x}$$

(a) Give the general solution of the reduced equation.

(b) Find a particular solution of the nonhomogeneous equation.

Answer: (a) $y = C_1e^{2x} + C_2xe^{2x}$ (b) $z = xe^{2x}\ln x$

17. Find the general solution of $y'' - \dfrac{4}{x}\,y' + \dfrac{6}{x^2}\,y = 4x$. HINT: The reduced equation has solutions of the form $y = x^r$.

Answer: $y = c_1x^2 + C_2x^3 + 4x^3\ln x$

18. Find a particular solution of $y'' + 4y = 2\tan\,2x$.

Answer: $z = -\frac{1}{2}\,\cos 2x\ln|\sec 2x + \tan 2x|$

19. The general solution of

$$y^{(4)} - 6\,y''' + 17\,y'' - 28\,y' + 20\,y = 0$$

is: (HINT: 2 is a root of the characteristic polynomial)

Answer: $y = C_1e^{2x} + C_2xe^{2x} + C_3e^{x}\cos 2x + C_4e^{x}\sin 2x$

20. Find the linear homogeneous equation of least order that has $y = 2xe^{-3x} + 4\cos 2x + 9x$ as a solution.

Answer: $y^{(6)} + 6y^{(5)} + 13y^{(4)} + 24y''' + 36y'' = 0$

21. Find the Laplace transform of the solution of the initial-value problem

$$y'' - 3y' - 6\,y = 2e^{2x} + 4;\;\; y(0) = 5,\;\; y'(0) = -2$$

Answer: $Y(s) = \dfrac{2}{(s-2)(s^2-3s-6)} + \dfrac{4}{s(s^2-3s-6)} + \dfrac{5s-17}{s^2-3s-6}$

22. Find $f(x) = \mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{3}{s^2} + \dfrac{4s+3}{s^2+4}$.

Answer: $f(x) = 3x + 4\cos 2x + \frac{3}{2}\sin 2x$

23. Find $f(x) = \mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{s^2 - 3s - 1}{(s-2)^2(s+4)}$

Answer: $f(x) = \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + \frac{3}{4}e^{-4x}$

24. Find $\mathcal{L}[f(x)]$ if

$$f(x) = \begin{cases} x^2 + 2x & 0 \le x < 4 \\ x & x \ge 4 \end{cases}$$

Answer: $F(s) = \dfrac{2}{s^3} + \dfrac{2}{s^2} - e^{-4s}\dfrac{2}{s^3} - 9e^{-4s}\dfrac{1}{s^2} - 20e^{-4s}\dfrac{1}{s}$

25. $F(s) = \dfrac{5}{s^2} - \dfrac{3}{s} - 2e^{-3s}\dfrac{1}{s^2} + 3e^{-3s}\dfrac{1}{s} + 2e^{-3s}\dfrac{s+1}{s^2+\pi^2}$. Find $\mathcal{L}^{-1}[F(s)] = f(x)$.

Answer: $f(x) = \begin{cases} 5x - 3, & 0 \le x < 3 \\ 3x + 6 - 2\cos \pi x - \frac{2}{\pi}\sin \pi x, & x \ge 3 \end{cases}$

26. Given the initial-value problem

$$y' - 4y = 2e^{-2x}, \; y(0) = 3.$$

(a) Find the Laplace transform of the solution.

(b) Find the solution by finding the inverse Laplace transform of your answer to (a).

Answer: (a) $Y(s) = \dfrac{2}{(s+2)(s-4)} + \dfrac{3}{s-4}$ (b) $y = \frac{10}{3}e^{4x} - \frac{1}{3}e^{-2x}$

27. Given the system of equations

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 5y - 4z &= 3 \\ -2x - 2y - 2z &= 0 \end{aligned}$$

(a) Write the augmented matrix for the system.

(b) Reduce the augmented matrix to row-echelon form.

(c) Give the solution set of the system.

Answer: (a) $\left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 2 & 5 & -4 & 3 \\ -2 & -2 & -2 & 0 \end{array}\right)$ (b) $\left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(c) $x = -1 - 3a,\; y = 1 + 2a,\; z = a$ a any real number

28. Determine the values of k so that the system of equations

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

has: (a) a unique solution, (b) no solutions, (c) infinitely many solutions

Answer: (a) $k \neq -3, 2$ (b) $k = -3$ (c) $k = 2$

29. Find the values of λ, (if any) such that $A = \begin{pmatrix} \lambda & 0 & 3 \\ 0 & 1 & \lambda \\ -2 & -1 & -5 \end{pmatrix}$ is nonsingular.

Answer: $\lambda \neq 2, 3$

30. The matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -3 \\ -2 & -4 & 1 \end{pmatrix}$ is nonsingular. Find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} -7 & -2 & -6 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

31. The system of equations

$$\begin{aligned} 2x - y + 3z &= 4 \\ y + 2z &= -2 \\ x + z &= 1 \end{aligned}$$

has a unique solution. Find y.

Answer: $y = -2$

32. Determine whether the vectors

$$\mathbf{v}_1 = (\,1,\ -3,\ 2\,),\ \mathbf{v}_2 = (\,0,\ -2,\ -2\,),$$

$$\mathbf{v}_3 = (\,1,\ -5,\ 0\,),\ \mathbf{v}_4 = (\,0,\ 4,\ 4\,)$$

are linearly dependent or linearly independent. If they are linearly dependent, find the maximal number of independent vectors.

Answer: The vectors are linearly dependent. The maximum number of independent vectors is 2.

33. For what values of a are the vectors

$$\mathbf{v}_1 = (\,1,\ 3,\ a\,),\ \mathbf{v}_2 = (\,-a,\ -2,\ -2\,),$$

$$\mathbf{v}_3 = (\,1,\ -1,\ 0\,)$$

linearly dependent?

Answer: $a = -4,\ 2$

34. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{pmatrix}$. Hint: 2 is an eigenvalue.

Answer: $\lambda_1 = 2,\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix};\quad \lambda_2 = -1,\ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix};\quad \lambda_3 = 1,\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

35. Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{pmatrix}$. Hint: 6 is an eigenvalue.

Answer: $\lambda_1 = 6,\ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix};\quad \lambda_2 = -2,\ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix};\quad \lambda_3 = -2,\ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

36. Find the eigenvalues and eigenvectors of $\begin{pmatrix} -2 & 1 & -1 \\ 3 & -3 & 4 \\ 3 & -1 & 2 \end{pmatrix}$. Hint: 1 is an eigenvalue.

Answer: $\lambda_1 = 1,\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\quad \lambda_2 = \lambda_3 = -2,\ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

37. Find the solution of the initial-value problem $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. HINT: 1 is an eigenvalue.

Answer: $\mathbf{x}(t) = 2e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - e^{t}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

38. Find a fundamental set of solutions of $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}$.

Answer: $\left\{ e^{3t}\left[\cos 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right],\ e^{3t}\left[\cos 2t\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \right\}$

39. The system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}\mathbf{x}$ is equivalent to a second order linear equation. (a) What is the equation? (b) Find the general solution of the system.

Answer: (a) $y'' - 6y' + 8y = 0$ (b) $\mathbf{x}(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

40. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\mathbf{x}$.

Answer: $\mathbf{x}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\left[e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + te^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$

41. Find the general solution of $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x}$. HINT: 2 is a root of the characteristic polynomial.

Answer: $\mathbf{x}(t) = C_1 e^{-t}\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3\left[e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + te^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right]$

42. Find a fundamental set of solutions of $\mathbf{x}' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}\mathbf{x}$. HINT: 10 is a root of the characteristic polynomial.

Answer: $\left\{e^t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},\ e^t\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\ e^{10t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right\}$