Summary: First Order Differential Equations

Given a first order differential equation

\[ y' = f(x, y) \]  \hspace{1cm} (*)

There are four types of equations that have solution methods.

1. **Linear differential equations.** Equation (*) is linear if it can be written in the form

   \[ y' + p(x)y = q(x) \]  \hspace{1cm} (standard form)

   **Solution method:** Multiply the equation by \( e^{\int p(x) \, dx} \) to obtain

   \[ e^{\int p(x) \, dx} y' + p(x)e^{\int p(x) \, dx} y = q(x)e^{\int p(x) \, dx} \]

   which is

   \[ \left(e^{\int p(x) \, dx} y\right)' = q(x)e^{\int p(x) \, dx} \]

   and integrate to find \( y \).

2. **Separable equations.** Equation (*) is separable if the function \( f \) can be factored as

   \[ f(x, y) = p(x)h(y) \]

   **Solution method:** Write the equation as

   \[ \frac{dy}{dx} = p(x)h(y) \]

   and divide by \( h(y) \) to obtain

   \[ \frac{1}{h(y)} \frac{dy}{dx} = p(x) \]

   or \[ q(y) \, dy = p(x) \, dx \] \quad \{q(y) = 1/h(y)\}

   Integrate to obtain \( Q(y) = P(x) + C \) and simplify.

3. **Bernoulli equations.** Equation (*) is a Bernoulli equation if it can be written in the form

   \[ y' + p(x)y = q(x)y^k \]  \hspace{1cm} (\( k \neq 0,1 \))

   **Solution method:** Divide the equation by \( y^k \) (i.e., multiply by \( y^{-k} \)) to obtain

   \[ y^{-k}y' + p(x)y^{1-k} = q(x) \]

   and change the variable by setting \( v = y^{1-k}, \ v' = (1-k)y^{-k}y' \). The resulting equation will be a linear equation in \( x \) and \( v \). Solve that equation to find \( v \). Finally, reverse the change of variable by replacing \( v \) by \( y^{1-k} \) and simplifying (if possible).

4. **Homogeneous equations.** Equation (*) is homogeneous if the function \( f \) satisfies

   \[ f(tx, ty) = f(x, y). \]

   **Solution method:** The change of variable \( y = vx, \ y' = v+vx' \) transforms the homogeneous equation (*) into the separable equation

   \[ \frac{1}{f(1,v)} - v \, dv = \frac{1}{x} \, dx. \]

   Solve this separable equation to obtain \( Q(v) = \ln x + C \), simplify as much as possible, and then replace \( v \) by \( y/x \) to revert to the original variables.
**Strategy for identifying a first order equation** \( y' = f(x, y) \)

**Step 1.** If \( y \) appears with power 1 only, then the equation might very well be linear. Write the equation in the standard form
\[
y' + p(x)y = q(x)
\]

**Step 2.** If \( y \) appears with powers other than 1, or if you cannot use Step 1, then attempt to factor \( f(x, y) \) into \( p(x)h(y) \). If \( f \) does factor, then the equation is separable.

**Step 3.** If Steps 1 and 2 fail, then the equation is either Bernoulli or homogeneous, or it is "none of the above."

(a) To check for Bernoulli, try to write the equation as \( y' + p(x)y = q(x)y^k \).

(b) Signals for homogeneous: If the equation contains a term such as \( e^{y/x} \), \( \sin(y/x) \), \( \cos(y/x) \), etc., then the equation is homogeneous; if \( f \) is an algebraic expression (e.g., quotient of powers of \( x \) and \( y \)), and all the terms have the same degree, then the equation is homogeneous.

**Beware** that an equation can be more than one type simultaneously, and that a given equation may be "none of the above."

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