Summary: First Order Differential Equations

Given a first order differential equation

$$y' = f(x, y) \qquad (*)$$

We have considered four types of first-order differential equarions equations that have solution methods:

1. **Linear differential equations.** Equation $(*)$ is linear if it can be written in the form

$$y' + p(x)y = q(x) \qquad \text{(standard form)}$$

Solution method: Multiply the equation by $e^{\int p(x)\,dx}$ to obtain

$$e^{\int p(x)\,dx}\,y' + p(x)e^{\int p(x)\,dx}\,y = q(x)\,e^{\int p(x)\,dx}$$

which is

$$\left(e^{\int p(x)\,dx}\,y\right)' = q(x)\,e^{\int p(x)\,dx}$$

and integrate to find y.

2. **Separable equations.** Equation $(*)$ is separable if the function f can be factored as

$$f(x, y) = p(x)h(y)$$

Solution method: Write the equation as

$$\frac{dy}{dx} = p(x)h(y)$$

and divide by $h(y)$ to obtain

$$\frac{1}{h(y)}\,dy = p(x)\,dx \qquad \text{or} \qquad q(y)\,dy = p(x)\,dx \quad [q(y) = 1/h(y)]$$

Integrate to obtain $Q(y) = P(x) + C$ and simplify.

3. **Bernoulli equations.** Equation $(*)$ is a Bernoulli equation if it can be written in the form

$$y' + p(x)y = q(x)y^k \qquad ((k \neq 0, 1)$$

Solution method: Divide the equation by y^k $\left(\text{i.e., multiply by } y^{-k}\right)$ to obtain

$$y^{-k}y' + p(x)y^{1-k} = q(x)$$

and change the variable by setting $v = y^{1-k}$, $v' = (1-k)y^{-k}y'$. The resulting equation will be a linear equation in x and v. Solve that equation to find v. Finally, reverse the change of variable by replacing v by y^{1-k} and simplifying (if possible).

4. **Homogeneous equations.** Equation $(*)$ is homogeneous if the function f satisfies

$$f(tx, ty) = f(x, y).$$

Solution method: The change of variable $y = vx,\ y' = v + xv'$ transforms the homogeneous equation $(*)$ into the separable equation

$$\frac{1}{f(1, v) - v}\, dv = \frac{1}{x}\, dx.$$

Solve this separable equation to obtain $Q(v) = \ln x + C$, simplify as much as possible, and then replace v by y/x to revert to the original variables.

Strategy for identifying a first order equation $y' = f(x, y)$

Step 1. If y appears with power 1 only, then the equation might be linear. It is if you can write the equation in the standard form

$$y' + p(x)y = q(x)$$

Step 2. If y appears with powers other than 1, or if you cannot use Step 1, then attempt to factor $f(x, y)$ into $p(x)h(y)$. If f does factor, then the equation is separable.

Step 3. If Steps 1 and 2 fail, then the equation is either Bernoulli or homogeneous, or it is "none of the above."

(a) To check for Bernoulli, try to write the equation as $y' + p(x)y = q(x)y^k$.

(b) Signals for homogeneous: If the equation contains a term such as $e^{y/x}$, $\sin(y/x)$, $\cos(y/x)$, etc., then the equation is homogeneous; if f is an algebraic expression (e.g., quotient of powers of x and y), and all the terms have the same degree, then the equation is homogeneous.

Beware that an equation can be more than one type simultaneously, and that a given equation may be "none of the above."