Summary: First Order Differential Equations

Given a first order differential equation

\[ y' = f(x, y) \]  

(\*)

There are four types of equations that have solution methods.

1. **Linear differential equations.** Equation (\*) is linear if it can be written in the form

\[ y' + p(x)y = q(x) \]  

(standard form)

**Solution method:** Multiply the equation by \( e^{\int p(x) \, dx} \) to obtain

\[ e^{\int p(x) \, dx} y' + p(x)e^{\int p(x) \, dx} y = q(x)e^{\int p(x) \, dx} \]

which is

\[ \left( e^{\int p(x) \, dx} y \right)' = q(x)e^{\int p(x) \, dx} \]

and integrate to find \( y \).

2. **Separable equations.** Equation (\*) is separable if the function \( f \) can be factored as

\[ f(x, y) = p(x)h(y) \]

**Solution method:** Write the equation as

\[ \frac{dy}{dx} = p(x)h(y) \]

and divide by \( h(y) \) to obtain

\[ \frac{1}{h(y)} \, dy = p(x) \, dx \quad \text{or} \quad q(y) \, dy = p(x) \, dx \quad [q(y) = 1/h(y)] \]

Integrate to obtain \( Q(y) = P(x) + C \) and simplify.

3. **Bernoulli equations.** Equation (\*) is a Bernoulli equation if it can be written in the form

\[ y' + p(x)y = q(x)y^k \]  

(\( k \neq 0, 1 \))

**Solution method:** Divide the equation by \( y^k \) (i.e., multiply by \( y^{-k} \)) to obtain

\[ y^{-k}y' + p(x)y^{1-k} = q(x) \]

and change the variable by setting \( v = y^{1-k}, \ v' = (1-k)y^{-k}y' \). The resulting equation will be a linear equation in \( x \) and \( v \). Solve that equation to find \( v \). Finally, reverse the change of variable by replacing \( v \) by \( y^{1-k} \) and simplifying (if possible).

4. **Homogeneous equations.** Equation (\*) is homogeneous if the function \( f \) satisfies

\[ f(tx, ty) = f(x, y). \]

**Solution method:** The change of variable \( y = vx, \ y' = v + xv' \) transforms the homogeneous equation (\*) into the separable equation

\[ \frac{1}{f(1, v)} \, dv = \frac{1}{x} \, dx. \]

Solve this separable equation to obtain \( Q(v) = \ln x + C \), simplify as much as possible, and then replace \( v \) by \( y/x \) to revert to the original variables.
Strategy for identifying a first order equation $y' = f(x, y)$

Step 1. If $y$ appears with power 1 only, then the equation might very well be linear. Write the equation in the standard form

$$y' + p(x)y = q(x)$$

Step 2. If $y$ appears with powers other than 1, or if you cannot use Step 1, then attempt to factor $f(x, y)$ into $p(x)h(y)$. If $f$ does factor, then the equation is separable.

Step 3. If Steps 1 and 2 fail, then the equation is either Bernoulli or homogeneous, or it is "none of the above."

(a) To check for Bernoulli, try to write the equation as $y' + p(x)y = q(x)y^k$.

(b) Signals for homogeneous: If the equation contains a term such as $e^{y/x}$, $\sin(y/x)$, $\cos(y/x)$, etc., then the equation is homogeneous; if $f$ is an algebraic expression (e.g., quotient of powers of $x$ and $y$), and all the terms have the same degree, then the equation is homogeneous.

Beware that an equation can be more than one type simultaneously, and that a given equation may be "none of the above."