Example: Bolza Problem
Consider the functional:
\[
\min \int_0^1 \left( \frac{u_t^2}{1 + u_t^2} + \frac{u_{tt}^2}{h(u)} \right) \, dx,
\]
subject to:
\[
\begin{align*}
0 &\leq u_t(x) \leq 1, \\
u(0) &= 0, \\
\end{align*}
\]
There are various ways to relax \( w(d) \):
- **Soft wall**
  \[
  w(d) = \begin{cases} 
  1 - \sqrt{\frac{d}{d^2 + 1}} & 0 \leq d \leq \sqrt{\frac{2}{h}} \\
  0 & \sqrt{\frac{2}{h}} < d \leq 1,
  \end{cases}
\]
- **Numerically using Young’s measures and method of moments.**

Extension: Gradient Flow
Given a Hilbert space, \( H \), with inner product \( \langle \cdot, \cdot \rangle \), the gradient flow of \( E(u) : H \to \mathbb{R} \) is a solution to the weak eq.
\[
(u_{n+1}, v) = -\langle \delta E(u), v \rangle = \min_{d \in \mathbb{R}} \frac{d}{\delta} \left( E(u + \delta v) - E(u) \right)
\]

Modified Split-Bregman Iteration

Recasting it as an unconstrained problem we get
\[
\min_{d \in \mathbb{R}} \langle \delta E(u), v \rangle = \min_{d \in \mathbb{R}} \left( E(u + \delta v) - E(u) \right)
\]

Modified Split-Bregman Iteration

Now, the original problem is \( \min \{ \Phi(u) + H(u) \} \), and this is equivalent to
\[
\min_{d \in \mathbb{R}} W(d) + H(u)
\]
Recasting it as an unconstrained problem we get
\[
\min_{d \in \mathbb{R}} W(d) + H(u) + \frac{1}{2} \| d - u \|_2^2
\]
and we may now use the Split-Bregman iteration:

1. \( d_{n+1} = \min_{d \in \mathbb{R}} W(d) + H(u)
\]
2. \( u_{n+1} = u_n + \alpha_s (d_{n+1} - d_n)
\]

Results: Bolza Problem
For the parameter values
\( \lambda = 2 \), \( h = 1/100 \), \( tol = 1e-8 \), \( \Delta x = 1/100 \),
our computations show:

**Method** | **N^t Iter.** | **Time Elapsed**
--- | --- | ---
Split Bregman | 210 | 0.413332
Split Bregman + G.F. | 171 | 0.406654

Figure: Solution computed with soft wall relaxation

Extensions: Non-Convex Potentials
If the functional is non-convex on the argument \( u \), as in this example:
\[
\min_{n \in \mathbb{R}} \int_0^1 \frac{1}{2} \left( -u_n + u_{n+1} + u_n^2 - u_{n+1}^2 + u_n^2 + u_{n+1}^2 \right) \, dx
\]
where in this case is
\[
\min_{n \in \mathbb{R}} \frac{1}{2} \left( -u_n + u_{n+1} + u_n^2 - u_{n+1}^2 + u_n^2 + u_{n+1}^2 \right)
\]
we can use convex splitting as follows:
\[
\min_{n \in \mathbb{R}} \left[ -u_n + u_{n+1} + u_n^2 - u_{n+1}^2 + u_n^2 + u_{n+1}^2 \right]
\]
and then adapt it for a modified Split-Bregman iteration
\[
\min_{n \in \mathbb{R}} \frac{1}{2} \left( -u_n + u_{n+1} + u_n^2 - u_{n+1}^2 + u_n^2 + u_{n+1}^2 \right)
\]
Future Work: Ribbon Problem
We are interested in understanding the different configurations of a ribbon subjected to tension and twist. These configurations can be found as minimizers of a functional involving the ribbon’s strain and bending energy. This functional is in general non-convex and depending on the parameterization used, may depend on two or more variables.

Ribbon Parametrisation and Energy

- **Parametrization for helicoidal shape**
  \[
  X(s,t) = ((1 - \chi) s, f(t) \cos(\alpha s), -g(t) \sin(\alpha s), f(t) \sin(\alpha s), g(t) \cos(\alpha s))
  \]
- **Strain Energy**
  \[
  E_s = \frac{E_0 L s}{S(1-\nu)} \int_0^1 \left( \left( 1 - \chi \right)^2 + \alpha_1 f' + \alpha_2 g' + \alpha_3 f'^2 + g'^2 \right) \, dt
  \]

Method of Moments [2]

Let \( f(x) \) be a polynomial of degree \( K \) with convex envelope \( f_e(x) \).

Figure: Convex envelope \( f_e(x) \)

Then there are numbers \( p_1, p_2 \) with \( p_1 + p_2 = 1 \), and points \( x_1, x_2 \) such that its graph
\[
(x, f(x)) = (p_1 x_1 + p_2 x_2, p_1 f(x_1) + p_2 f(x_2))
\]
We can think of \( \mu = p_1 \delta_{x_1} + p_2 \delta_{x_2} \) as probability measure. Then
\[
\int_{\mathbb{R}} f(x) \, d\mu = \min_{n \in \mathbb{R}} \int_{\mathbb{R}} f(x) \, d\mu(\lambda)
\]
with \( \Omega \) the set of probability measures with mean \( x \).

Truncated Moment Problem

From Curto and Faiklow [3], given a vector in \( \mathbb{R}^n \), it is possible to find a discrete probability measure supported in \( \mathbb{R} \) with moments equal to the values of said vector.

Numerical Computation of \( f(x) \):

For the parameter values
\( \lambda = 2 \), \( h = 1/100 \), \( tol = 1e-8 \), \( \Delta x = 1/100 \),
our computations show:

**Method** | **N^t Iter.** | **Time Elapsed**
--- | --- | ---
Split Bregman | 210 | 0.413332
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Figure: Relaxation of \( (d^2 - 1)^2 \)

**Bibliography**