Summary of Complex Functions and Mappings

A function $f(z)$ is analytic in a region D if it is differentiable at every point in the region. This is quite restrictive as it means that if $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$, then the functions u, v satisfy the Cauchy Riemann equations.

A function that is analytic for all $z \in \mathbb{C}$ is said to be entire. Otherwise it is either not defined everywhere in $\mathbb{C}$ or it has **singularities**.

When a function has an **isolated singularity** that is a pole of order m, there are formulae for evaluating line integrals on closed contours around the singulariiy (or many singularities) This involves the **residue at each singularity**.

The calculus of residues provides **formulae for contour integrals** when a function has singularities inside a closed contour.

There is a way to interpret ∞ geometrically as a point at the north pole of a sphere.

When f(z) is analytic in a region D, you often need to know whether $f(z) = c$ has one or more solutions in D.

This may be studied geometrically by looking at the mapping properties of $f(z)$.

You can count the number of solutions (and singlarities) in a region by evaluating a contour integral around a curve that encloses the region

The crucial formulae and results are

1. Cauchy's integral theorem.
2. Cauchy's integral formula
3 Generalized Cauchy's integral formula.
4. Fundamental theorem of algebra and the formulae for the number of solutions of an equation inside a contour.
5. Representations of functions by Taylor and Laurent series. What is a singularity and what is the order of a pole or a zero.
6. Representations of functions by mappings of the z-plane to the w-plane. Can say when a function has a zero or a pole at infinity.
7. The residue theorem; applications to evaluate trigonometric integrals, Fourier and Laplace transforms.

Most of the functions that arise in physics / science are, or can be approximated by, analytic functions. So there is a need to know about their singularities, zeroes, derivaives, integrals, Taylor series, ...

Also want to know about various transforms, series and special approximations. Very often if you cannot solve a problem in calculus I easily, then you should look at it as a complex variable problem and see if you can obtain an answer using analytic function theory

A topic that is used very often in complex analysis is changing contours of integration. The rules are;

1. Suppose Γ_1, Γ_2 and two contours from a to b in $\mathbb{C}$ and D is the region between them. If $f(z)$ is analytic in D then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

2. Suppose C_1, C_2 are two simple closed loops in the plane that do not intersect, and D is the region between them. If $f(z)$ is analytic on D, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

so choose your contours and loops to make the calculation easy!

Complex Mappings and Functions

A complex function defined on a domain D can be visualized geometrically as a mapping of the plane to itself. Suppose $f : D(\subset \mathbb{C}) \to \mathbb{C}$ is an analytic function. Then we know that

$$f(z) \;=\; u(x,y) \;+\; i\,v(x,y)$$

can be represented by its real and imaginary parts. For example

$$z^2 \;=\; (x^2 - y^2) \;+\; 2ixy \quad \text{and} \quad z^3 \;=\; (x^3 - 3xy^2) \;+\; i(3x^2y - y^3)$$

$$\text{Write} \qquad F(x,y) \;:=\; \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

We often graph the contour lines of u, v. That is the curves in the plane where

$$u(x, y) = c, \quad v(x, y) = d \quad \text{for different values of c,d}$$

The contour lines for z^2 are hyperbolae, $x^2 - y^2 = c$ and $xy = d$. The derivative of $f(z)$ is represented by a matrix.

$$f'(z) \ = \ DF(x, y) \ = \ \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

This matrix has $|f'(z)|^2 \ = \ det(DF(x, y))$ and it is singular if and only if $f'(z) = 0$.

$DF(x,y)$ is called the Jacobian matrix of F at (x,y) and its determinant $det(DF(x,y))$ is called the Jacobian of F at (x,y).

A function (map) $f : D \to \mathbb{C}$ is said to be **1-1 on D** provided $z, \zeta \in D$ and $z \neq \zeta$ implies that $f(z) \neq f(\zeta)$. Let $G = f(D)$, and f be 1-1 on D, then there is an inverse function $g : G \to D$ such that

$$g(w) \;:=\; z \quad \text{when} \quad w \;=\; f(z)$$

For example if $f : B \to \mathbb{C}$ is defined on a ball B of radius less than $\pi/2$, and $f(z) := \sin z$, then its inverse is $g(w) \;:=\; arcsin(w)$.

The main theorem about inverse functions is the following. It is much the same as the theorem in 1-d calculus.

Theorem (Inverse function) Suppose that f is analytic at a point $z_0 \in D$ and $f'(z_0) \neq 0$. Then there is an open disk B_r centered at z_0 and a function $g : G \rightarrow B_r$ that is inverse to f with $g(w_0) = z_0$. Moreover g is an analytic function of w near w_0.

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When γ_1, γ_2 are two differentiable curves that intersect at a point $z_0 \in D$, then the angle between the curves is θ where θ is the angle between the unit tangent vectors to the curves, so

$$\cos\theta \ := \ t_1 \cdot t_2$$

If this angle is zero the curves are said to be tangent (kiss) while if $\theta = \pi/2$, they are **orthogonal**.

Note that the level curves of the functions u, v are orthogonal at all points where $f'(z_0) \neq 0$. Thus the level curves can be used to define a grid on a set D. The usual rectangular grid is given by the level curves of $f(z) = z$.

When γ_1, γ_2 are two differentiable curves in D that intersect at a point z_0 then their images under an analytic function f will again be differentiable curves defined by

$$\sigma_1 := \{f(\gamma_1(t)) : t \in I\} \quad \text{and} \quad \sigma_2 := \{f(\gamma_2(t)) : t \in I\}$$

that will intersect at $w_0 := f(z_0)$.

A mapping $F : D \to R^2$ is said to be **conformal** if it maps two differentiable curves γ_1, γ_2 in D into differentiable curves σ_1, σ_2 in R^2 with the property that if γ_1, γ_2 intersect at an angle θ in D then σ_1, σ_2 also intersect at the angle θ in R^2.

The first theorem about complex maps is

Theorem. Suppose that f is analytic at a point $z_0 \in D$ with $f'(z_0) \neq 0$ then the associated map $F : D \to R^2$ is a conformal map on a neighborhood of (x_0, y_0).

So any map defined by the real and imaginary part of an analytic function is 1-1 and conformal near every point where its Jacobian is non-zero.

This is an essential property for many maps used in navigation, flying etc. You need the angles in the map to be correct The Mercator projection is a conformal projection.

You know that a straight line in the plane of slope m has that equation $y = mx + b$ in Cartesian coordinates. What is its equation in the form $z = \zeta(t)$?

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You know that a straight line in the plane of slope m has that equation $y = mx + b$ in Cartesian coordinates. What is its equation in the form $z = \zeta(t)$?

Ans: $\zeta(t) = t + i(mt + b) \quad -\infty < t < \infty$

The equation of a circle is $(x-a)^2+(y-b)^2=r^2$ in Cartesian coordinates. Let $z_0=a+ib$, then the complex equation is $|z-z_0|^2=r^2$. Alternatively it has the parametric equation $\zeta(\theta) \;=\; z_0+r\,e^{i\theta}$ for $-\pi \le \theta \le \pi$. Consider the function

$$f(z) \;:=\; \frac{az+b}{cz+d} \quad \text{for } z \neq -d/c$$

If $c=0,\; d\neq 0$ this is a linear transformationn that maps lines to lines and circles to circles. When $c\neq 0$ this defines a linear fractional transformation by $w=f(z)$. .

Assume $ad-bc\neq 0$, then this function maps every complex number z, except (-d/c) onto another complex number w.

This mapping is 1-1 from $D\backslash\{-d/c\}$ onto $\mathbb{C}$ and its inverse is another linear fractional transformation.

A domain D is **proper** if $\mathbb{C} \backslash D \neq \emptyset$. B_1 is the unit disk.
Theorem (Riemann Mapping) Let D be a proper simply conneted domain in $\mathbb{C}$. Then there is a 1-1 analytic function that maps D onto B_1.

It took almost fifty years from when Riemann stated this result before a (reasonably) rigorous proof was given.

How to find this function? There are formulae / algorithms (the Schwarz-Christoffel construction) for mapping a polygon onto the unit disk. See appendix A of the text.

In general you must solve a problem in the calculus of variations. According to Wikipedia this is an important problem in image processing and there still is research on the numercial constructions of these mappings.