Summary of Complex Functions and Mappings

A function $f(z)$ is analytic in a region $D$ if it is differentiable at every point in the region. This is quite restrictive as it means that if $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$, then the functions $u, v$ satisfy the Cauchy Riemann equations.

A function that is analytic for all $z \in \mathbb{C}$ is said to be entire. Otherwise it is either not defined everywhere in $\mathbb{C}$ or it has singularities.

When a function has an isolated singularity that is a pole of order $m$, there are formulae for evaluating line integrals on closed contours around the singularities (or many singularities) This involves the residue at each singularity.

The calculus of residues provides formulae for contour integrals when a function has singularities inside a closed contour.
There is a way to interpret $\infty$ geometrically as a point at the north pole of a sphere.

When $f(z)$ is analytic in a region $D$, you often need to know whether $f(z) = c$ has one or more solutions in $D$.

This may be studied geometrically by looking at the mapping properties of $f(z)$.

You can count the number of solutions (and singularities) in a region by evaluating a contour integral around a curve that encloses the region.
The crucial formulae and results are
1. Cauchy’s integral theorem.
2. Cauchy’s integral formula
4. Fundamental theorem of algebra and the formulae for the number of solutions of an equation inside a contour.
5. Representations of functions by Taylor and Laurent series. What is a singularity and what is the order of a pole or a zero.
6. Representations of functions by mappings of the z-plane to the w-plane. Can say when a function has a zero or a pole at infinity.
7. The residue theorem; applications to evaluate trigonometric integrals, Fourier and Laplace transforms.
Most of the functions that arise in physics/ science are, or can be approximated by, analytic functions. So there is a need to know about their singularities, zeroes, derivatives, integrals, Taylor series, ...

Also want to know about various transforms, series and special approximations. Very often if you cannot solve a problem in calculus I easily, then you should look at it as a complex variable problem and see if you can obtain an answer using analytic function theory
A topic that is used very often in complex analysis is changing contours of integration. The rules are;

1. Suppose $\Gamma_1, \Gamma_2$ and two contours from $a$ to $b$ in $\mathbb{C}$ and $D$ is the region between them. If $f(z)$ is analytic in $D$ then

$$\int_{\Gamma_1} f(z) \, dz = \int_{\Gamma_2} f(z) \, dz$$

2. Suppose $C_1, C_2$ are two simple closed loops in the plane that do not intersect, and $D$ is the region between them. If $f(z)$ is analytic on $D$, then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz$$

so choose your contours and loops to make the calculation easy!
Complex Mappings and Functions

A complex function defined on a domain $D$ can be visualized geometrically as a mapping of the plane to itself. Suppose $f : D(\subset \mathbb{C}) \rightarrow \mathbb{C}$ is an analytic function. Then we know that

$$f(z) = u(x, y) + iv(x, y)$$

can be represented by its real and imaginary parts. For example

$$z^2 = (x^2 - y^2) + 2ixy \quad \text{and} \quad z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Write $F(x, y) := \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$
We often graph the contour lines of $u, v$. That is the curves in the plane where

$$u(x, y) = c, \quad v(x, y) = d$$

for different values of $c, d$

The contour lines for $z^2$ are hyperbolae, $x^2 - y^2 = c$ and $xy = d$. The derivative of $f(z)$ is represented by a matrix.

$$f'(z) = DF(x, y) = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}$$

This matrix has $|f'(z)|^2 = det(DF(x, y))$ and it is singular if and only if $f'(z) = 0$. 
$DF(x, y)$ is called the Jacobian matrix of $F$ at $(x, y)$ and its determinant $\text{det}(DF(x, y))$ is called the Jacobian of $F$ at $(x, y)$.

A function (map) $f: D \to \mathbb{C}$ is said to be **1-1 on $D$** provided $z, \zeta \in D$ and $z \neq \zeta$ implies that $f(z) \neq f(\zeta)$. Let $G = f(D)$, and if $f$ be 1-1 on $D$, then there is an inverse function $g: G \to D$ such that

$$g(w) := z \text{ when } w = f(z)$$

For example if $f: B \to \mathbb{C}$ is defined on a ball $B$ of radius less than $\pi/2$, and $f(z) := \sin z$, then its inverse is $g(w) := \text{arcsin}(w)$.
The main theorem about inverse functions is the following. It is much the same as the theorem in 1-d calculus.

**Theorem (Inverse function)** Suppose that $f$ is analytic at a point $z_0 \in D$ and $f'(z_0) \neq 0$. Then there is an open disk $B_r$ centered at $z_0$ and a function $g : G \to B_r$ that is inverse to $f$ with $g(w_0) = z_0$. Moreover $g$ is an analytic function of $w$ near $w_0$. 
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When $\gamma_1, \gamma_2$ are two differentiable curves that intersect at a point $z_0 \in D$, then the angle between the curves is $\theta$ where $\theta$ is the angle between the unit tangent vectors to the curves, so

$$\cos \theta := t_1 \cdot t_2$$

If this angle is zero the curves are said to be tangent (kiss) while if $\theta = \pi/2$, they are **orthogonal**.
Note that the level curves of the functions $u, v$ are orthogonal at all points where $f'(z_0) \neq 0$. Thus the level curves can be used to define a grid on a set $D$. The usual rectangular grid is given by the level curves of $f(z) = z$.

When $\gamma_1, \gamma_2$ are two differentiable curves in $D$ that intersect at a point $z_0$ then their images under an analytic function $f$ will again be differentiable curves defined by

$$
\sigma_1 := \{ f(\gamma_1(t)) : t \in I \} \quad \text{and} \quad \sigma_2 := \{ f(\gamma_2(t)) : t \in I \}
$$

that will intersect at $w_0 := f(z_0)$. 
A mapping \( F : D \to \mathbb{R}^2 \) is said to be **conformal** if it maps two differentiable curves \( \gamma_1, \gamma_2 \) in \( D \) into differentiable curves \( \sigma_1, \sigma_2 \) in \( \mathbb{R}^2 \) with the property that if \( \gamma_1, \gamma_2 \) intersect at an angle \( \theta \) in \( D \) then \( \sigma_1, \sigma_2 \) also intersect at the angle \( \theta \) in \( \mathbb{R}^2 \).

The first theorem about complex maps is

**Theorem.** Suppose that \( f \) is analytic at a point \( z_0 \in D \) with \( f'(z_0) \neq 0 \) then the associated map \( F : D \to \mathbb{R}^2 \) is a conformal map on a neighborhood of \((x_0, y_0)\).

So any map defined by the real and imaginary part of an analytic function is 1-1 and conformal near every point where its Jacobian is non-zero.
This is an essential property for many maps used in navigation, flying etc. You need the angles in the map to be correct. The Mercator projection is a conformal projection.

You know that a straight line in the plane of slope $m$ has that equation $y = mx + b$ in Cartesian coordinates. What is its equation in the form $z = \zeta(t)$?
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You know that a straight line in the plane of slope \( m \) has that equation \( y = mx + b \) in Cartesian coordinates. What is its equation in the form \( z = \zeta(t) \)?

Ans: \( \zeta(t) = t + i(mt + b) \quad -\infty < t < \infty \)
The equation of a circle is \((x - a)^2 + (y - b)^2 = r^2\) in Cartesian coordinates. Let \(z_0 = a + ib\), then the complex equation is \(|z - z_0|^2 = r^2\). Alternatively it has the parametric equation \(\zeta(\theta) = z_0 + r e^{i\theta}\) for \(-\pi \leq \theta \leq \pi\). Consider the function

\[
f(z) := \frac{az + b}{cz + d} \quad \text{for } z \neq -d/c
\]

If \(c = 0, \ d \neq 0\) this is a linear transformation that maps lines to lines and circles to circles. When \(c \neq 0\) this defines a linear fractional transformation by \(w = f(z)\).

Assume \(ad - bc \neq 0\), then this function maps every complex number \(z\), except \((-d/c)\) onto another complex number \(w\).

This mapping is 1-1 from \(D\{−d/c\}\) onto \(\mathbb{C}\) and its inverse is another linear fractional transformation.
A domain $D$ is **proper** if $\mathbb{C} \setminus D \neq \emptyset$. $B_1$ is the unit disk.

**Theorem (Riemann Mapping)**  Let $D$ be a proper simply connected domain in $\mathbb{C}$. Then there is a 1-1 analytic function that maps $D$ onto $B_1$.

It took almost fifty years from when Riemann stated this result before a (reasonably) rigorous proof was given.

How to find this function? There are formulae / algorithms (the Schwarz-Christoffel construction) for mapping a polygon onto the unit disk. See appendix A of the text.

In general you must solve a problem in the calculus of variations. According to Wikipedia this is an important problem in image processing and there still is research on the numerical constructions of these mappings.