Sequences of Complex Numbers

We'll now discuss questions of convergence and approximation of complex numbers and functions. Some of this theory is similar to the theory of real functions of a real variable x but the results often are much nicer.

In general a **sequence** in a set S is an infinite set $\{s_m : m \geq 1\}$ of points in S indexed by the positive integers. They need not all be different. Here we shall usually just consider sequences of real or complex numbers or functions.

The sequence $\mathcal{S} := \{z_m : m \geq 1\}$ is a sequence of complex numbers when each z_m is a complex number. Usually we are interested in sequences that converge to a specific (complex) number. The sequence is said to **converge to a limit** $\tilde{z}$ provided that for any choice of $\epsilon > 0$, there is an $M(\epsilon)$ such that

$$m > M(\epsilon) \;\Rightarrow\; |z_m - \tilde{z}| < \epsilon.$$

That is every z_m wih m large enough is within a distance ϵ of the limit $\tilde{z}$. When a limit in this sense exists, it must be unique. (You cannot have two different limits $\tilde{z}_1, \tilde{z}_2$.

When there is no such $\tilde{z}$, then the sequence is said **diverge.**

Often sequences are defined explicitly by an iterative algorithm such as the linear difference equation

$$z_{m+1} \;=\; a\, z_m + b, \qquad z_0 = 1,\; m \geq 0$$

Here a,b are complex numbers. Using elementary methods you can show that such a sequence will converge to a finite limit when $|a| < 1$ and will not converge when $|a| > 1$.

If this sequence converges, then it converges to the limit $\tilde{z} \;=\; b/(1-a)$. So the sequence definitely does not converge when $a = 1$. (Why?, evaluate the first 4 or so terms and find an explicit formula for z_m when a $= 1$.)

The sequence defined by $s_{m+1} = s_m + a_m, \; s_0 = 0$ for $m \geq 0$ has

$$s_{m+1} = a_0 + a_m + \ldots + a_m = \sum_{j=0}^{m} a_j$$

If this sequence has a limit as $m \to \infty$, then we have

$$\tilde{s} := \lim_{m \to \infty} s_m = \sum_{j=0}^{\infty} a_j$$

and $s_m, \tilde{s}$ arerespectively called the partial sums, and the sum of this series. The series is said to be divergent when these partial sums do not converge to a limit. In this case the infinite sum is not a well defined mathematical quantity.

When f is a continuous function on $\mathbb{C}$ consider the sequence defined by $z_{m+1} := f(z_m)$ for $m \geq 0$ and z_0 given.

If z_m converges to a limit $\tilde{z}$ as $m \to \infty$, then it is easy to show that $\tilde{z}$ is a fixed point of f. That is, $\tilde{z}$ is a solution of $z = f(z)$. (Why?)

The sequence

$$z_{m+1} := \frac{1}{2}\left(z_m + \frac{a}{z_m}\right) \quad m = 1, 2, 3, \ldots$$

converges to one of the two numbers $\pm\sqrt{a}$ when $a, z_1 \in \mathbb{C}$. (Why?) Which one it converges to depends on where you start; the choice of z_1.

If $a = 3,\ z_0 = 1$, then you obtain

$$1,\ 2,\ \frac{7}{4},\ \frac{97}{56} = 1.73214286, \ldots$$

A calculator gives $\sqrt{3} = 1.732050808\ldots$. Suggest that you try this algorithm with $a = i, -1, -4$ or some other favorite complex number.

When you know the possible limits, as in this problem, it is usually easy to see whether a particular sequence converges to the limit - or not. This will depend on the choice of z_0.

An example of a sequence of partial sums is the sequence given by

$$s_1 = 1, \quad s_{m+1} := s_m + \frac{(-1)^{m+1}}{2m-1} \quad m \geq 1$$

This is a sequence of rational numbers that converges very slowly to the number $\pi/4$. It takes M $= 200$ to have the answer to 2 decimal places and 5 billion iterations to obtain π to 10 decimal places. For example

$$s_7 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$$

It is not hard to prove that the sequence converges but it is not a practical way for estimating π.

There are many "tests" for the convergence of complex series. The **comparison test** says that a series $\sum_{j=0}^{\infty} c_j$ converges if there is a sequence of positive real numbers M_j such that $|c_j| \leq M_j$ for all j and $\sum_{j=0}^{J} M_j$ converges to a limiting real number as $J \to \infty$. Note that the sums of the partial sums with M_j is an increasing sequence of real numbers so this sequence has a limit if and only if it is bounded above by a real number S.

The **ratio test** says that the series $\sum_{j=0}^{\infty} c_j$ converges if

$$\lim_{j \to \infty} \left| \frac{c_{j+1}}{c_j} \right| = L < 1.$$

When $L > 1$ then the series diverges.

There are many other tests for sequences to converge that may be found in texts - but usually one wants to describe not only whether it converges but how fast it converges. In numerical computations, we often talk about the rate of convergence of a sequence of numbers to a limit. This is a matter of describing how $M(\epsilon)$ grows as $\epsilon \to 0$. You would like this to be a power law of the form $M(\epsilon) \leq C \epsilon^{-d}$ for some C, d positive.

A number z_m is said to be an approximation that is accurate to d decimal places provided $|z_m - \tilde{z}| < 5 \times 10^{-(d+1)}$.

The sequence defined by $z_m := 1 + \sum_{j=1}^{m} \frac{1}{j!}$ is a sequence of rational numbers that converges very rapidly to the transcendental number e since $m!$ grows very rapidly as m increases.

Sequences and Series of Analytic Functions

A function is said to be **analytic** on a domain D if every derivative $f^{(k)}(z)$ exists and is continuous on D. From the Cauchy integral formulae it is enough that it just be differentiable on D. Thus from now on in this course we shall use the adjective **analytic** rather than **differentiable** for complex functions and use the fact that all the derivatives exist and are diffferentiable on D.

Let D be a non-empty domain in $\mathbb{C}$ and $\{f_m : m \geq 1\}$ be a sequence of analytic functions on D. This sequence converges to a function $f(z)$ on some subset D_1 of D provided the sequences of complex numbers $\{f_m(z) : m \geq 1\}$ converges to $f(z)$ for each z in D_1.

Geometric Sums and Series.

The geometric sum of degree M is the polynomial

$$s_M(z) \ := \ 1 + z + z^2 + \ldots + z^M.$$

This function is analytic for all $z \in \mathbb{C}$. Special values include $s_M(0) = 1, \ s_M(1) = M + 1$ for all integers M. In high school you may have seen a proof that

$$s_M(z) \ := \ \frac{z^{M+1} \ - \ 1}{z \ - \ 1} \qquad \text{for } z \in \mathbb{C}, \ z \neq 1$$

If not, please verify.

What happens as $M \to \infty$?

When $|z| \geq 1$, then this sum has no limit as $M \to \infty$. Look at what happens for $z = \pm 1, \pm i$ and see that in these cases the sum takes just a couple of values but doesn't have a limit.

When $|z| = r < 1$ then $|z|^M \to 0$ as $M \to \infty$ and the sequence of polynomials converges to the function $F(z) := 1/(1-z)$ for all $z \in B_1(0)$ and not for any z with $|z| > 1$.

Note that this convergence of a sequence of functions only occurs on a proper subset of the domain $\mathbb{C}$.

Suggest that you verify this computationally. Find out how large M must be for $|r^M| \leq 0.5 \times 10^{-5}$ for $r = 0.2, 0.4, 0.5, 0.7, 0.9$? Find a formula for this M.

Most complex analysis texts have a chapter on the convergence of sequences of polynomials and in particular, the convergence of power series.

These are sequences defined by

$$s_m(z) \;:=\; s_{m-1}(z) \;+\; c_m\,(z - z_0)^m \quad \text{for } m \geq 1$$

and $s_0(z) \;=\; c_0$ with z_0 and the c_m all complex numbers. Thus

$$s_m(z) \;=\; \sum_{j=0}^{m} \; c_j\,(z - z_0)^j$$

is a polynomial of degree m.

If only a finite number of the c_m are non-zero, this limit will be a polynomial. and analytic on $\mathbb{C}$

Note that $s_m(z_0) = c_0$ for all m so this sequence will converge at z_0. From the difference equation, one sees that if the sequence $\{s_m(z)\}$ converges to a limit at a point $z \neq z_0$ then you must have $\lim_{m\to\infty} |c_m (z - z_0)^m| = 0$.

In your first calculus course you should have learnt about Taylor polynomials, Taylor approximations of functions and Taylor series. These have especially nice properties when the functions is an analytic function on a domain $D \subset \mathbb{C}$.

Taylor approximations of degree m

When $f : D \to \mathbb{C}$ is an analytic function, $z_0 \in D$ then the linear approximation to $f(z)$ near z_0 is the affine function

$$T_1 f(z) \ := \ f(z_0) \ + \ f'(z_0)\,(z - z_0)$$

The quadratic approximation is defined by

$$T_2 f(z) \ := \ f(z_0) \ + \ f'(z_0)\,(z - z_0) \ + \ \frac{f''(z_0)}{2}\,(z - z_0)^2$$

and for each integer m, the m-th Taylor approximation is

$$T_m f(z) \ := \ T_{m-1} f(z) \ + \ \frac{f^{(m)}(z_0)}{m!}\,(z - z_0)^m.$$

The m-th Taylor approximation is a polynomial in $(z - z_0)$ of degree less than or equal to m. When $f(z)$ is a polynomial of degree M, then all its derivatives of order k with $k \geq M + 1$ are identically 0, so this becomes a formula for the polynomial in terms of powers of $(z - z_0)$.

When f is analytic on D, then the Taylor series of f at z is the expression

$$Tf(z) := \lim_{m \to \infty} T_m f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m.$$

provided this limit exists. This is a formal definition so a question is when does this limit exist as a differentiable function of z near z_0?

Ex: $f(z) := Log(1+z)$ is analytic on a disk of radius 1 centered at z=0 , with $Log(1) = 0$. Then

$$T_3 f(z) = z \; - \; \frac{z^2}{2} \; + \; \frac{z^3}{3}$$

and the Taylor series of $Log(1+z)$ is

$$\sum_{m=1}^{\infty} \frac{(-z)^m}{(m)!} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$$

Each of the Taylor approximations $T_m f(z)$ is a polynomial but this Taylor series will only converge to $Log(1+z)$ when $|z| < 1$. It does not converge when $z = -1$. The sequence of approximations has $T_m f(-1) \to -\infty$ as $m \to \infty$. Note that when $z = 2$, then the series does not converge to $Log\, 3 = 1.0986$

A Taylor series about $z_0 = 0$ is often called a Maclaurin series. The main theorem about the convergence of Taylor series is the following nice result.

Theorem. Suppose that $f(z)$ is analytic on the disk $B_R(z_0)$, then the Taylor polynomials $T_m f(z)$ converge to $f(z)$ as $m \to \infty$ for all $z \in B_R(z_0)$.

Thus Taylor series converges to the actual function f(z) on any disk center z_0 on which $f(z)$ is analytic, That is on disks where $f(z)$ is differentiable. The preceding example of $Log(1 + z)$ shows that the Taylor series of an analytic function need not always converge to the function at points where the function is analytic, but the following does hold.

Corollary. Suppose that $f(z)$ is a differentiable on $\mathbb{C}$, then the Taylor polynomials $T_m f(z)$ converge to $f(z)$ as $m \to \infty$ for all $z \in \mathbb{C}$.

Examples of Maclaurin Series include

1. $$e^z = 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$$

2. $$\sin z = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^{2j-1}}{(2j-1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots.$$

3. $$\cos z = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots.$$

These come from the formulae for the derivatives and their value at $z_0 = 0$. Since these functions are analytic on $\mathbb{C}$ these series converge for every $z \in \mathbb{C}$. Similarly for $cosh(z)$, $sinh(z)$.

The Maclaurin series for $tan(z)$ can be found but only converges for $|z| < \pi/2$.

Example. Find the Taylor series of $Log(z)$ about the point $z_0 = 2$. The function $Log(z)$ has derivatives $1/z, -z^{-2}, 2\,z^{-3}, \ldots$ and in general the m-th derivative is $(-1)^{m-1}\,(m-1)!\;z^{-m}$. If $z = x + iy$, then $(z - z_0) = (x-2) + iy$ and the Taylor series has m-th term

$$\frac{f^{(m)}(2)}{m!}\;(z-2)^m$$

. Thus

$$Log(z) \;=\; \ln 2 \;+\; \frac{(z-2)}{2} \;-\; \frac{(z-2)^2}{8} \;+\; \frac{(z-2)^3}{48} + \ldots$$

This is a power series in $(z-2)$ and uses the fact that $Log(2) = \ln 2$ and all the derivatives of the function $Log(z)$ are real when $z_0 = x_0$ is a positive number.

There are other possible formulae for the coefficients in a Taylor series (or expansion.) From the generalized Cauchy formula one has

$$a_m \;=\; \frac{f^{(m)}(z_0)}{m!} \;=\; \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}}\,dz$$

where C is a simple closed contour around z_0 with $f(z)$ analytic inside C. Usually C is chosen to be a circle center z_0 and of a small radius.

The proof of the result about convergence of the Taylor series on a disk of radius R about z_0 when the function is analytic on that disk, uses this expression and the generalized Cauchy integral formulae. There also are integral formulae for the error term in approximating an analytic function by a Taylor polynomial of the specific order $m \geq 1$.

The other major result is that if $T_m(f)$ is the Taylor polynomial of order m that approximates $f(z)$ near $z = z_0$, and f is differentiable on a disk $B_R(z_0)$, then the Taylor polynomial of degree (m-1) that approximates $f'(z)$ near z_0 is the derivative of $T_m(f)$. That is you can differentiate term by term and the new series converges. It says that if

$$f(z) \;=\; c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ... + c_m\,(z - z_0)^m + ...$$

and this series converges, then

$$f'(z) \;=\; c_1 + 2c_2(z - z_0) + .. + mc_m(z - z_0)^{m-1} + ...$$

and this series converges.

The above theorems say that if we know the values of an analytic function and all of its derivatives at a point z_0, then we know the function in a disk center at that point - provided the Taylor approximations converge to a function there.

In general there is a value R called the radius of convergence of this series such that the series

(i) converges in the disk $|z - z_0| < R$, and

(ii) does not converge when $|z - z_0| > R$.

R is called the radius of convergence and could be 0. It will be ∞ for an entire function. It can be proved that, if these limits exist,

$$R = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}, \quad \text{or}$$

$$R^{-1} = \lim_{n \to \infty} |a_n|^{1/n}$$

using the ratio test or the root test respectively.

Another very useful result is the following. Suppose that a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges at a point $z_1 \neq z_0$ with $R_1 = |z_1 - z_0|$.

Theorem Suppose this power series converges at z_1 and $R_1 = |z_1 - z_0|$. Then the series converges at every point z obeying $|z - z_0| < R_1$ and the sum is an analytic function on this disk of radius R_1.

Example. The function $f(z) = \frac{z}{1+z^2}$ has the Maclaurin series about $z = 0$ given by

$$f(z) = z - z^3 + z^5 - z^7 + \ldots .$$

You can verify that this series gives the value of the function at $z = 1/2$ to 3 decimal places with about 5 terms of the series.

Find the radius of convergence of this series and evaluate the partial sums at $z = \pm 1, \pm i$. You could also find its Taylor series about any other $z_0 \neq \pm i$. Try $z_0 = \pm 1$.

Integral Tests for convergence of series

Sometimes a simple method for finding whether a series converges is to use a comparison test with specific integrals. Consider

$$S \ := \ \sum_{n=1}^{\infty} a_n \quad \text{or} \quad S(z) \ := \ \sum_{n=1}^{\infty} a_n(z)$$

Suppose that there is a positive decreasing function $\varphi(t)$ defined for $1 \leq t < \infty$ that interpolates the values of $|a_n|$ or $|a_n(z)|$. That is $\varphi(n) = |a_n(z)|$. Then from the definition of a Riemann integral

$$\sum_{n=1}^{\infty} a_{n+1} \ \leq \ \int_1^{\infty} \varphi(t)\, dt \ \leq \ \sum_{n=1}^{\infty} a_n$$

So if you can evaluate this integral you have an lower and upper bounds on the sum of these series.

Zeroes of Complex Functions

Suppose D is a domain and $f : D \to \mathbb{C}$ is an analytic function. If $z_0 \in D$ is a point with $f(z_0) \ = \ 0$, then z_0 is said to be a **zero of f on D.** z_0 is said to be a **simple zero of f** on D if $f'(z_0) \neq 0$.

When z_0 is a simple zero then the Taylor series of f is

$$f(z) \ = \ \sum_{k=1}^{\infty} a_k \, (z - z_0)^k \ = \ (z - z_0) \sum_{k=1}^{\infty} a_k \, (z - z_0)^{k-1}$$

where the coefficients a_k are the Taylor coefficients and $a_1 \neq 0$. That is $f(z) \ = \ (z - z_0) \, f_1(z)$ for z near z_0 and $f_1(z)$ is an analytic function on a disk $B_R(z_0)$ and $f_1(z_0) \neq 0$. This is called a factorization of f near z_0.

When f is a polynomial of degree M, then f_1 will be a polynomial of degree $M - 1$.

More generally z_0 is a **zero of order m** provided $f^{(k)}(z_0) = 0$ for $0 \le k \le m-1$ and $f^{(m)}(z_0) \neq 0$. In this case the Taylor series of f is given by

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k = (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$$

where the coefficients a_m are the Taylor coefficients, so $a_m \neq 0$. That yields the factoziation $f(z) = (z - z_0)^m f_1(z)$ for z near z_0 with $f_1(z)$ an analytic function on a disk $B_R(z_0)$ and $f_1(z_0) \neq 0$. When f is a polynomial of degree M, then f_1 will be a polynomial of degree $M - m$ and z_0 is said to be a zero of multiplicity m of f.

When every derivative of f at z_0 is zero then the Taylor series of $f(z)$ will be identically zero on any disk centered at z_0, so $f(z) \equiv 0$ on this disk.

Singularities of Complex Functions

Suppose $f(z)$ is analytic on a domain $D_0 \ := \ D \setminus \{z_0\}$ where $z_0 \in D$. Then z_0 is a **singularity of f** of D if the limit of $f(z)$ as $z \to z_0$ and $z \in D_0$ does not exist. Examples include

1. $$f(z) \ := \ \frac{C}{(z \ - \ z_0)^4} \quad \text{with } C \neq 0,$$

2. $f(z) \ := \ \frac{p(z)}{q(z)}$ where p, q are polynomials in z and $z_0 \in \mathbb{C}$ with $q(z_0) = 0, \ p(z_0) \neq 0$.

3. $f(z) \ := \ exp(-1/z^2)$ for $z \neq 0$.

Suppose z_0 is an isolated singularity of an analytic function $f(z)$, then

(i) z_0 is a **pole of order k** for $f(z)$ provided there is a nonzero complex number b_k and a $k \in \mathbb{N}$ such that

$$\lim_{z \to z_0} (z - z_0)^k f(z) \; = \; b_k \quad (\neq 0).$$

(ii) z_0 is an **essential singularity** of $f(z)$ when it is a singular point but these limits do not exist for any integer k.

The preceding example 1 has a pole of order 4 at z_0. The second example has a pole of order m at z_0 when z_0 is a zero of order m of q and $p(z_0) \neq 0$.

The function $f(z) \ := \ exp(-1/z^2)$ for $z \neq 0$ has an essential singularity at the origin. From the series for the exponential function, this is given formally by

$$f(z) \ = \ 1 - \frac{1}{z^2} + \frac{1}{2\,z^4} - \frac{1}{3!\,z^6} - \frac{1}{4!\,z^8} + \ldots$$

Sometimes a function is said to have a removable singularity if it is defined by a formula that appears to make $f(z_0)$ bad, but $\lim_{z \to z_0} \ f(z) \ = \ c$ is finite. In this case make sure that $f(z_0) = c$.

We have seen that when z_0 is a zero of order M of an analytic function f(z), then for z near z_0, the Taylor series can be factored to have a simple form. A similar result holds near poles of order k of a function.

Laurent Series

Suppose $f(z)$ is an analytic function on an annular domain $A := \{z : r_1 < |z - z_0| < r_2\}$ (If $r_1 = 0$ this is called a deleted disk; when $r_1 > 0$ this is a open annulus.)

In this case the function can be written as the sum of two power series, one in powers of $(z - z_0)$ and the other in powers of $(z - z_0)^{-1}$. Then

$$f(z) \ = \ \sum_{k=-\infty}^{\infty} c_k(z \ - \ z_0)^k \quad \text{for } z \in A$$

This series converges everywhere in A with the c_k being

$$c_k \ := \ \frac{1}{2\pi i} \int_C \ \frac{f(z)}{(z - z_0)^{k+1}} \ dz$$

Here k is any positive or negative integer and C is any **simple closed contour in A that goes around z_0.**

Consider the complex function $f(z) := z^{-1}$. This function has a simple pole at $z = 0$ with a 1-term Laurent series around the origin $z = 0$.

For any $z_0 \neq 0$, this is analytic function on the ball $B_R(z_0)$ with $R < |z_0|$. It has derivatives

$$\frac{d^k f}{dz^k}(z) \ = \ (-1)^k \ \frac{k!}{z^{k+1}}$$

so there is a convergent Taylor series for $1/z$ around every $z_0 \neq 0$. (Write it out!).

Similarly consider the finite geometric sum

$$f(z) \ := \ 2 + \frac{1}{z} + \frac{1}{z^2} + \ldots + \frac{1}{z^m}$$

This function has a pole at $z = 0$ and you can verify that

$$f(z) = \frac{2z^{m+1} - z^m - 1}{z^{m+1} - z^m}$$

Here the denominator is

$$z^{m+1} - z^m \ = \ z^m \ (z - 1).$$

so you may think that both 0 and 1 are poles. However $f(1) = m + 2$ so 1 is not a pole, and f(z) is analytic near $z = 1$ so this function has a Taylor series in powers of (z-1).

The functions cosec z, $\cot z$ have simple poles at $z = 0$ with Laurent series representations in deleted disks around 0 with $R < \pi$.

Complex Infinity

The function $f(z) = 1/z$ maps circles of radius $r(> 0)$ around the origin into circles of radius $1/r$. In particular it maps the positively oriented unit circle into a negatively oriented unit circle. It maps the positive (or negative) real axis into itself, but the positive imaginary axis into the negative imaginary axis. (Please verify these claims!)

Thus $w = f(z) \;=\; z^{-1}$ is 1-1 and onto from $\mathbb{C}\backslash\{0\}$ to itself. The inverse function is $z = 1/w$ so $f^{-1}(w) = f(w)$ for all $w \in \mathbb{C}\backslash\{0\}$. As a consequence we **define** limits of functions as $z \to \infty$ by

$$\lim_{z\to\infty} f(z) \;:=\; \lim_{w\to 0} f(1/w) \quad \text{whenever this RHS exists.}$$

Example. Consider the rational function $f_k(z) = z^k/(1+z^2)$. This function has poles at $z = \pm i$, is zero at the origin and is analytic elsewhere in $\mathbb{C}$. It is bounded on the real axis when $k = 0, 1, 2$ and unbounded when $k \geq 3$. You can show that

$$\lim_{z \to \infty} f_k(z) = 0 \quad \text{for } k = 0, 1 \text{ and } \quad 1 \quad \text{when } k = 2.$$

In particular we say that $f(z)$ is analytic near ∞ provided $g(w) = f(1/w)$ is analytic near $w = 0$.

$f(z)$ has a zero of order m at infinity if $g(w)$ has a zero of order m at $w = 0$.

$f(z)$ has a pole of order m at infinity if $g(w)$ has a pole of order m at $w = 0$.

$f(z)$ has an essential singularity at infinity if $g(w)$ has an essential singularity at 0.

Examples 1. The function defined by $f(z) = (z-a)^{-m}$ has a zero of order m at ∞.

2. The exponential function (or $\sinh z, \cosh z, \sin z$) have essential singularities at ∞.

3. What can you say about the behavior at infinity of the function f_k above with $k \geq 2$?

The Riemann Sphere and Infinity

Geometrically the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ may be visualized as points on the **Riemann sphere**. This is a sphere in 3d with center $(0, 0, 1/2)$ and radius $1/2$. The north pole is at $(0, 0, 1)$ and the south pole at the origin.

Draw the straight lines from the north pole through the sphere and identify points on the sphere with the complex number $z = x + iy$ - the point where the straight line intersects the horizontal plane $x_3 = 0$. Points in the upper hemisphere are related to points outside the unit disk, points on the equator are mapped to points on the unit circle in $\mathbb{C}$ and points in the southern hemisphere are mapped to points inside the unit disk.

The point at ∞ will be identified with the North pole. This is called the stereographic projection. Since this provides a 1-1 mapping of $\overline{\mathbb{C}}$ with this sphere, we say that the sets are isomorphic under this mapping (or function).

The Riemann Zeta Function

The Riemann zeta function is defined for $Re(z) > 1$ by the series

$$\zeta(z) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n^z}$$

When $z = s + iy$ then $|n^{-z}| \;=\; e^{-s\ln n} \;=\; n^{-s}$. (Check this out.) Then the integral inequality yields

$$|\zeta(z)| \;\leq\; \sum_{n=1}^{\infty} n^{-s} \;\leq\; 1 \;+\; \frac{1}{s-1} = \frac{s}{s-1}$$

when $Re(z) = s > 1$. When $s = 1$ the series diverges.

This function is one of the most studied functions in mathematics.

In other undergraduate classes, you may see proofs that $\zeta(2) = \frac{\pi^2}{6}$ and various formulae for the sum. In particular $\zeta(3) = 1.20205\ldots$ is called Apery's constant who proved in 1978 that it is an irrational number. (See Wikipedia).

The function has a pole at $z = 1$ and can be defined (using different formulae) for all other complex numbers. (The same function is given by different formulae in different domains).

The **Riemann hypthesis** (1859) is still an open problem with a one million dollar $++$ prize for whoever proves or disproves it. It asks whether all the non-trivial zeroes of $\zeta(z)$ lie on the line $\Re(z) = 1/2$? See Wikipedia for a long article about the problem that describe many different results about the zeroes. Evaluating this function is a major test problem for computers and, now, networks.