Complex Numbers and Functions

In the first three weeks of the class we have covered much of the material in Chapters 1 through 3 of the textbook. You should understand and be able to use

1. Complex numbers and complex arithmetic, polar form, $|z|$, arg z.
2. The definitions of various types of subsets of $\mathbb{C}$, (open, closed, connected, bounded, boundary domain and region).
3. Limits and convergence of a sequence, continuity of a function, rules for sums, products , quotients and compositions.
4. Differentiability of a function, Cauchy-Riemann equations and harmonic functions. Harmoic conjugate function.
5. Complex Polynomials, Exponential, Trigonometric and Hyperbolic functions.
6. Roots of unity, Complex Logarithms and Powers.

In particular chapter 2 of the text concentrated on the definition and properties of differentiable (= analytic) complex functions. Chapter 3 studied the exponential function and showed that - in terms of complex z - the exponential, trigonometric and hyperbolic functions all are related.

Most of this theory is quite similar to the theory of 2-dimensional calculus. Chapter 4 of the text book treats the complex integration. Here we will treat integrals that are line integrals in the complex plane and the theory is quite unlike what holds in previous courses.

First we need the definitions for complex integration. Then these definitions will be used to obtain formulae and results that are used throughout science and engineering.

Using complex inetgrals, many new analytic functions such as Bessel functions, Legendre functions, rational functions, gamma and zeta functions are defined and their properties established.

The definitive online source for this is Digital Library of Mathematical Functions at dlmf.nist.gov/4.2 Essentially all the material in the first 25 sections (chapters) involves analytic functions; also sections 28-33. DLMF is the online successor to the **Handbook of Mathematical Functions,** edited by Abramowitz and Stegun, and published in 1964, which was for more than 50 years one of the most cited books in science. It is over 1000 pages long and has always been legally available for free download on the web.

Integrals of complex functions

When $f(t)$ is a complex valued function of a real variable with $f(t) = u(t) + iv(t)$, then the integral

$$\int_a^b f(t)\,dt \;:=\; \int_a^b u(t)\,dt \;+\; i\int_a^b v(t)\,dt$$

is a complex number whose real and complex parts are the integrals you had in Calculus I.

They have the same properties as ordinary integrals with, in particular,

$$\left|\int_a^b f(t)\,dt\right| \;\le\; \int_a^b |f(t)|\,dt$$

Complex Integration

Chapter 4 of the text is about complex integration. The main results in this chapter are Cauchy's integral theorem which says that a lot of complex (line) integrals are zero and Cauchy's integral formula which says that $f(z_0)$ is equal to a special integral of $f(z)$ around z_0.

This leads to many theorems about analytic functions - including useful formulae and inequalities - and the fundamental theorem of algebra. The integrals are primarily **contour integrals** of the form

$$\int_\Gamma f(z)\,dz \qquad \text{where } \Gamma \text{ is a } \textbf{piecewise smooth curve}$$

in the complex plane and $f(z)$ is a complex function defined on Γ. This integral should be a complex number.

How to define this integral? In elementary integration, you evaluate integrals over an interval (a, b). In 2d calculus there are integrals over domains D in the plane with respect to area $dxdy$. Contour integrals are similar to line integrals from calculus 3.

A **curve** (or arc) in the complex plane is the range of a complex function defined on an open or closed interval $I \subset \mathbb{R}$. That is if $\zeta : I \to \mathbb{C}$ is a continuous function then

$$\Gamma \;:=\; \{\, \zeta(t) : \; t \in I \,\}.$$

When $I = [a, b]$ is a closed interval then $\zeta(a), \zeta(b)$ are called the **end-points** of the curve. If $\zeta(a) = \zeta(b)$, then Γ is said to be a **loop** or a **closed curve**. Occasionally a single point z_0 is called a loop - defined by the function $\zeta : [0, 1] \to \mathbb{C}$ with $\zeta(t) = z_0$ for all t.

When a smooth curve is parametrized by a function ζ then the **arc-length** of the curve can be evaluated. The distance of a point $\zeta(t) = x(t) + iy(t)$ from the initial point $z_0 = \zeta(a)$ is

$$s(t) \;:=\; \int_a^t \;|\dot{\zeta}(\tau)|\; d\tau.$$

This $s(t)$ is called the arc-length of $\zeta(t)$ from z_0 along Γ and

$$\frac{ds}{dt}(t) \;=\; |\dot{\zeta}(t)| = \sqrt{\dot{x}(t)^2 \;+\; \dot{y}(t)^2} \;>\; 0$$

The length of Γ is

$$|\Gamma| \;=\; L(\Gamma) \;:=\; \int_a^b \;|\dot{\zeta}(t)|\; dt.$$

Example: An ellipse center at the origin and with semi-axes a, b has the parametric form

$$\zeta(\theta) \ := \ a \cos\theta + i\, b \sin\theta \quad 0 \leq \theta \leq 2\pi$$

This is a simple loop The length of the arc of this ellipse between $z_0 = a$ and $z_1 = i\, b$ corresponds to the arc going from $\theta = 0$ to $\theta = \pi/2$ so has length

$$L(\Gamma) \ := \ \int_0^{\pi/2} \ \sqrt{a^2 \sin^2\theta \ + \ b^2 \cos^2\theta}\ d\theta.$$

The perimeter of the ellipse is 4 times this value and the area inside the ellipse is $\pi\, a\, b$. This is an **elliptic integral** which does not have a simple expression in terms of functions you have seen when $a \neq b$

Suppose that we choose a grid $G := \{z_0, z_1, \ldots, z_M\}$ of consecutive points on Γ from $z_0 = \zeta(a)$ to $z_M = \zeta(b)$. Then

$$I_G(f) := \sum_{j=1}^{M-1} f(\zeta_j)\,(z_{j+1} - z_j) \quad \text{with } \zeta_j \in (z_j, z_{j+1})$$

is a Riemann sum for this integral contour integral. Let $h_G := \max_j |z_{j+1} - z_j|$ be the grid mesh. Then

$$I_\Gamma(f) := \int_\Gamma f(z)\,dz \; := \; \lim_{h_G \to 0} \sum_{j=0}^{M-1} f(\zeta_j)\,(z_{j+1} - z_j)$$

This limit should be a complex number.

A curve Γ is self-intersecting if there are distinct values t_1, t_2 in I that are not both end points and such that $\zeta(t_1) = \zeta(t_2)$. If a curve is not self-intersecting then the function ζ is 1-1 - except possibly at the end-points if it is a loop.

The curve Γ is said to be a **directed smooth curve (or arc)** when it is defined by a function $\zeta : [a, b] \to \mathbb{C}$ and

(i) Γ is not self-intersecting,

(ii) $\dot{\zeta}(t)$ is non-zero, finite and continuous on I := [a,b].

Here $\dot{\zeta}(t) := \dot{x}(t) + i\dot{y}(t)$ is the usual (1-d) derivative with respect to t. In particular a curve has a direction; it goes from $\zeta(a)$ to $\zeta(b)$. ζ provides a **parametrization** of Γ. The same curve can have different parametrizations.

When f, g are continuous complex functions defined on a directed smooth curve (contour) Γ of finite length, the contour integral $\int_\Gamma (f)$ will be a complex number. They obey the usual rules including Linearity

$$\int_\Gamma [c_1 f_1(z) + c_2 f_2(z)]\, dz \;=\; \int_\Gamma c_1 f_1(z)\, dz + \int_\Gamma c_2 f_2(z)\, dz$$

Triangle inequality: If $|f(z)| \leq M$ on Γ, then

$$\left| \int_\Gamma f(z)\, dz \right| \;\leq\; M\; L(\Gamma).$$

Orientation: Let $-\Gamma$ be the curve in the reverse direction from $\zeta(b)$ to $\zeta(a)$. Then

$$\int_{-\Gamma} f(z)\, dz \;=\; -\int_\Gamma f(z)\, dz$$

Example: Let $z_0 \in \mathbb{C}$ and C be the circle of center z_0 and radius r. Find the integrals $\int_C (z - z_0)^n \, dz$ with $n \in \mathbb{Z}$.

First choose a parametrization of C such as $\zeta(\theta) := z_0 + r\, e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. This is a regular arc and a loop. On C, $f(\zeta(\theta)) = r^n \, e^{in\theta}$. Thus

$$\int_C (z - z_0)^n \, dz = \int_0^{2\pi} i r^{n+1} \, e^{i(n+1)\theta} \, d\theta$$

When $n \neq -1$, this integral with respect to θ is zero. When $n = -1$ one finds that

$$\int_C \frac{1}{z - z_0} \, dz = 2\pi i$$

so this integral is independent of r!

Very often we wish to work with curves that are not smooth curves - but have "corners". For example triangles, rectangles, polygons or similar sets.

Suppose we have a finite number of smooth curves $\gamma_1, \ldots, \gamma_n$ where the initial point of the component γ_{j+1} is the final point of the component γ_j for $1 \leq j \leq n-1$.

Then $\Gamma := (\gamma_1, \ldots, \gamma_n)$ is called a **contour** from the initial point of γ_1 to the final point of γ_n. In particular a contour has an orientation and $-\Gamma$ is the same set but with the reverse ordering. All the preceding rules for evaluating integrals hold when contours are used in place of arcs.

Above we showed that the integral of a power function like $(z-\zeta)^m$ is zero around any circle centered at ζ when $m \neq -1$. It turns out that such integrals are very often zero when **the function f(z) is analytic around** Γ.

Cauchy -Goursat Theorem Suppose a function $f(z)$ is analytic on and inside a simple closed contour C. then

$$\int_C f(z)\, dz = 0$$

Suppose that a function $f(z)$ is analytic in a region D, z_0, z_1 are two points in D and C_1, C_2 are two curves from z_0 to z_1 that lie in D. Then the curve obtained by joining $-C_2$ to C_1 (or -C_1 to C_2) is a close loop in D. So

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

Another way of saying this is that when $f(z)$ is analytic in a region D, z_0, z are points in the region, then the integral

$$F(z) \ := \ \int_{z_0}^{z} f(z)\, dz \ = \ \int_C \ f(z) \ dz$$

for any curve C that goes from z_0 to z and remains in D. That is this indefinite contour integral is independent of the path from z_0 to z!

This integral may be defined for all $z \in D$ so F is a complex valued function on D. The text has a problem that asks you to show that this function is differentiable (or analytic) on D and that

$$F'(z) \ = \ \frac{dF}{dz}(z) \ = \ f(z) \quad \text{for all } z \in D.$$

A simple loop is positively oriented if you go around it clockwise. It is negatively oriented when you go around it anticlockwise.

The boundary of a domain $D \subset \mathbb{C}$ consists of some closed loops and possibly some isolated points. When the domain D consists of all points inside a simple closed loop, D is said to be **simply connected** or not have any "holes".

Suppose D is the set of points inside a simple closed loop C_0 but outside a finite number of simple closed loops $C_1, \ldots, C_J$ (that don't intesect each other or C_0). Then the domain D is said to be multiply connected and the boundary has J+1 connected components. In his case the domain has J "holes".

Many results about complex integrals in this class will be stated only for integrals in domains that are simply connected. There are associated formulae for contour integrals in multiply connected regions but they need some rules about "winding numbers".

Cauchy's Integral Formula

Let D be the simply connected domain inside a simple closed positively oriented loop C, If f is analytic on domain D, then the values of f in D are given by a contour integral. Namely

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz \qquad \text{for } z_0 \in D$$

That is, the value of an analytic function at a point z_0 inside a loop C can be found by knowng the value of this integral along a contour around the point. When one knows the values of an analytic function on a circle or a rectangle or a general polygon in the complex plane, then this Integral formula gives the value of the function at any point inside the loop.

Generalized Cauchy's Integral Formula

Let D be the simply connected domain inside a simple closed positively oriented loop C, If f is analytic on domain D, then the values of the k-th derivative $f^{(k)}(z)$ at a point z in D is given by a contour integral. Namely

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}}\, dz \qquad \text{for } z_0 \in D$$

Since each of these integrals can be evaluated and is finite, this formula is the basis of the usual proof that if $f(z)$ is analytic inside and on the loop C then all its derivatives exist and are finite at points inside C.

The result that is used to prove the Generalized Cauchy Integral Formula is that if Γ is a regular arc of finite length, and a complex function g is continuous on the arc, then the function

$$G(z) \ := \ \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

is analytic at any point $z \notin \gamma$. Moreover

$$G'(z) \ = \ \frac{dG}{dz}(z) \ = \ \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$$

$$G''(z) \ = \ \frac{d^2 G}{dz^2}(z) \ = \ 2 \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^3} d\zeta$$

. . . and all these integrals are complex numbers. This is called "differentiation under the integral sign".

Examples Evaluate the following integrals around a closed loop C of your choice. Say what loops will make the integrals zero - or not.

$$\int_C \frac{\sin z}{z}\, dz \qquad \int_C \frac{\sin z}{2z - \pi}\, dz$$

$$\int_C \frac{z^3 + z - 1}{(2z-4)^2}\, dz \qquad \int_C \frac{e^{-2z}}{z^4}\, dz$$

When you see integrals of fractions where there are polynomials or other simple functions in the denominator, look to see where this denoiminator is zero. These are the "singularities" and integrals on loops around the singular points may be non zero.

Bounds on Functions and Derivatives.

The importance of the Cauchy integral formulae is that when we know an analytic function on a contour Γ then you can evaluate it at any point inside the contour - by evaluating a contour integral. This allows one to also find bounds on the values of the function.

Suppose that a function is analytic on the circle C_R center z_0 and radius R. Suppose that $|f(\zeta)| \leq M$ for $\zeta \in C_R$. Then $|f(z)| \leq M$ inside C_R and Cauchy's integral formula says that

$$|f'(z_0)| \leq \frac{M}{R} \qquad \text{and}$$

$$|\, f^{(k)}(z_0)\,| \leq \frac{M\,k!}{R^k}.$$

A function f on $\mathbb{C}$ is said to be **entire** provided it is analytic for all z.

Theorem (Liouville) If an entire function is bounded on $\mathbb{C}$, then it is constant on $\mathbb{C}$.

Suppose now that $p(z)$ is a polynomial of degree n. That is

$$p(z) \ := \ a_0 z^n \ + \ a_1 z^{n-1} \ + \ \ldots + a_{n-1} z \ + \ a_n$$

with $a_0 \neq 0$ and each $a_j \in \mathbb{C}$. Then

$$\lim_{|z| \to \infty} \frac{p(z)}{z^n} \ = \ a_0 .$$

A basic question in algebra has been how do you find the zeros of $p(z)$. That is to find the solutions of $p(z) \ = \ 0$.

Are there formulae for these zeroes? Yes for $n = 2, 3, 4$. No for $n \geq 5$.

The Fundamental Theorem of Algebra.

Theorem A complex polynomial of degree n has a complex zero.

Suppose ζ_1 is a complex zero, then

$$p(z) \;=\; (z-\zeta_1)(a_0 z^{n-1} \;+\; b_1 z^{n-2} + \ldots + b_{n-2} z + b_{n-1}$$

where you can evaluate the new coefficients b_j. Try to find the formulae for them!

Thus $p(z) \;=\; (z-\zeta_1)q_1(z)$ with $q(z)$ a polynomial of degree $(n-1)$ and leading coefficient $a_0 \neq 0$. Apply the theorem again to find a second zero ζ_2 and $p(z)$ is the product of two factors $(z-\zeta_1)(z-\zeta_2)\,q_2(z)$ with q_2 a polynomial of degree $(n-2)$. Continue until you have a product of n simple factors $(z-\zeta_j)$ - so the ζ_j will be the zeroes of the polynomial $p(z)$. This is called factoring a polynomial.

A zero ζ_j of $p(z)$ has multiplicity m if $p(z) = (z - \zeta_j)^m q(z)$ with $q(\zeta_j) \neq 0$. Then there will be at most n distinct complex zeroes of $p(z)$ and the number of zeros counting multiplicity is n.

Example Factor $p(z) = z^4 - 2z^3 + 2z^2 - 2z + 1$ and determine the number of distinct zeros and their multiplicity.

A sequence is an infinite set $\{z_m : m \geq 1\}$ indexed by the positive integers. They are sequences of complex numbers if each z_m is a complex number.

Usually we are interested in sequences that converge to a specific (complex) number. For example you can show that for most choices of z_1 the sequence defined by

$$z_{m+1} \ := \ \frac{1}{2}\left(z_m + \frac{a}{z_m}\right) \qquad m = 1, 2, 3, \ldots$$

converges to one of the two numbers $\pm\sqrt{a}$ when $a \in \mathbb{C}$.

Example If $a = 3, \ z_0 = 1$, then you obtain

$$1, 2, \frac{7}{4}, \ \frac{97}{56} = 1.73214286, \ldots$$

A calculator gives $\sqrt{3} = 1.732050808\ldots$.

This algorithm holds even for complex numbers such as finding the square root of $a := 1 + i \; = \; \sqrt{2}e^{i\pi/4}$. You find an infnite sequence of complex numbers that will converge to $\sqrt{a}$ from $z_0 = 1$ or $1 + i$.

Similarlly the set of numbers defined by

$$z_1 = 1, \quad z_{m+1} := z_m + \frac{(-1)^{m+1}}{2m-1} \quad m \geq 1$$

defines a sequence of rational numbers that converges very slowly to the number $\pi/4$. It takes M $= 200$ to have the answer to 2 decimal places and 5 billion iterations to obtain π to 10 decimal places. Here

$$z_7 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$$

The sequence is said to converge to a limit ζ provided that for any choice of $\epsilon > 0$, there is an $M(\epsilon)$ such that

$$m > M(\epsilon) \;\Rightarrow\; |z_m - \zeta| < \epsilon.$$

That is every z_m wih m large enough is within a distance ϵ of the limit ζ.

Often take $\epsilon = 10^{-d}$ and this M tells you how many terms are required to obtain numbers that are correct to d decimal places.

In complex analysis we are mostly interested in limits of polynomials and functions.

Geometric Sums and Series.

The geometric sum of degree M is the expression

$$s_M(z) \;:=\; 1 + z + z^2 + \ldots + z^M.$$

This is computable with $s_m(0) = 1, \; s_M(1) = M + 1$ and

$$s_M(z) \;:=\; \frac{z^{M+1} \;-\; 1}{z \;-\; 1} \qquad \text{for } z \neq 1$$

What happens as $M \to \infty$? When $|z| \geq 1$, then this sum has no limit as $M \to \infty$

Observe that if $|z| = r < 1$ then $|z|^M \to 0$ as $M \to \infty$.
Check this out computationally. Find out how large M must be for $|r^M| \;\leq\; 0.5 \;\times 10^{-5}$ **for** $r = 0.2, 0.4, 0.5, 0.7, 0.9$? Find a formula for this M.

In general when $\{c_0, c_1, \ldots, c_m, \ldots\}$ is an infinite sequence of complex numbers, then

$$s_m \;:=\; \sum_{j=0}^{m} c_j \;=\; c_0 + c_1 + \ldots + c_m$$

is called the m-th partial sum of these numbers. If the sequence $\{s_0, s_1, \ldots, s_m, \ldots\}$ of complex numbers converges to a limit $S \in \mathbb{C}$, then S is called the sum of the infinite series. We write

$$S \;:=\; \lim_{m\to\infty} \; s_m \;=\; \sum_{j=0}^{\infty} \; c_j$$

Examples include

$$e \;=\; 1 \;+\; \sum_{j=1}^{\infty} \; \frac{1}{j!}$$

This series converges very quickly as $j!$ grows very rapidly.

There are many "tests" for the convergence of complex series.

The **comparison test** says that the series $\sum_{j=0}^{\infty} c_j$ converges provided there is a sequence of positive real numbers M_j such that $|c_j| \leq M_j$ for all j and $\sum_{j=0}^{\infty} M_j$ converges.

The **ratio test** says that the series $\sum_{j=0}^{\infty} c_j$ converges if

$$\lim_{j\to\infty} \left| \frac{c_{j+1}}{c_j} \right| = L < 1.$$

If $L > 1$ then the series diverges.

In your first calculus course you should have heard about Taylor approximations of functions and Taylor series. These have especially good properties when the functions is an analytic function on a domain $D \subset \mathbb{C}$.

Suppose $z_0 \in D$ then the linear approximation to $f(z)$ near z_0 is the linear function

$$T_1 f(z) \; := \; f(z_0) \; + \; f'(z_0)\,(z - z_0)$$

The quadratic approximation is

$$T_2 f(z) \; := \; f(z_0) \; + \; f'(z_0)\,(z - z_0) \; + \; \frac{f''(z_0)}{2}\,(z - z_0)^2$$

In general the m-th Taylor approximation is the polynomial of degree m defined by

$$T_m f(z) \; := \; T_{m-1} f(z_0) \; + \; \frac{f^{(m)}(z_0)}{m!}\,(z - z_0)^m.$$

When f is analytic on D, then it is infinitely differentiable at each point in D and the Taylor series of f at given by

$$Tf(z) := \lim_{m\to\infty} T_m f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m.$$

Ex: $f(z) := Log(1+z)$ is analytic on a disk of radius 1 centered at z=0 , with $Log(1) = 0$. Then

$$T_3 f(z) = z - \frac{z^2}{2} + \frac{z^3}{3}$$

and the Taylor series of $Log(1+z)$ is

$$\sum_{m=1}^{\infty} \frac{(-z)^m}{(m)!} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

Each of the Taylor approximations $T_m f(z)$ is an entire function on $\mathbb{C}$ but the Taylor series will only converge to $Log(1+z)$ when $|z| < 1$. It does not converge when $z = -1$. The sequence of approximations has $T_m f(-1) \to -\infty$ as $m \to \infty$

Theorem. Suppose that $f(z)$ is analytic on the disk $B_R(z_0)$, then the Taylor polynomials $T_m f(z)$ converge to $f(z)$ as $m \to \infty$ for all $z \in B_R(z_0)$.

Corollary. Suppose that $f(z)$ is an entire function, then the Taylor polynomials $T_m f(z)$ converge to $f(z)$ as $m \to \infty$ for all $z \in \mathbb{C}$.

Examples of Maclaurin Series include

1.
$$e^z = 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!}.$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$$

2.
$$\sin z = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^{2j-1}}{(2j-1)!}.$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots.$$

3.
$$\cos z = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}.$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots.$$

Since these functions are entire these series converge for every $z \in \mathbb{C}$. Similarly for $cosh(z)$, $sinh(z)$. The Maclaurin series for $tan(z)$ can be found but only converges for $|z| < \pi/2$.

General Power Series

Given an infinite sequence of complex numbers $\{a_0, a_1, \ldots, a_j, \ldots\}$, and $z_0 \in \mathbb{C}$ the infinite series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a **power series** (in $(z - z_0)$ or about z_0).

When only a finite number of the a_n are non-zero this is a polynomial. If there is a number $\delta > 0$ and infinitely many a_n with $|a_j| \geq \delta$ then this series does not converge for any $z \neq z_0$.

So a necessary condition for a series to converge is that $\lim_{n\to\infty} |a_n| = 0$.

In the text of Kwok, he states results about absolute convergence - which is stronger than convergence. In this class we will not work with functions that are convergent but not absolutely convergent or be concenred with the difference.

Suppose that a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges at a point $z_1 \neq z_0$ with $R_1 = |z_1 - z_0|$.

Theorem Suppose the series converges at z_1 and $R_1 = |z_1 - z_0|$. Then the series converges at every point z obeying $|z - z_0| < R_1$ and the sum is an analytic function on this disk of radius R_1.

In general there is a value R called the radius of convergence of this series such that the series

(i) converges in the disk $|z - z_0| < R$, and

(ii) does not converge when $|z - z_0| > R$.

R is called the radius of convergence and could be 0. It will be ∞ for an entire function. It can be proved that, if these limits exist,

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}, \quad \text{or}$$

$$R^{-1} = \lim_{n\to\infty} |a_n|^{1/n}$$

using the ratio test or the root test respectively.

This shows that power series in $(z - z_0)$ define analytic functions that converge on disks or radius R around z_0. Conversely when $f(z)$ is analytic on a disk near z_0, its Taylor series will converge inside the disk. That is

$$f(z) \;=\; \lim_{N\to\infty} \sum_{n=0}^{N} a_n\,(z - z_0)^n, \text{ when } |z - z_0| < R$$

and $a_n := f^{(n)}(z_0)/n!$. A Maclaurin series is a Taylor series with $z_0 = 0$. The function $f(z) \;=\; \frac{z}{1+z^2}$ has the Maclaurin series about $z = 0$ given by

$$f(z) \;=\; z - z^3 + z^5 - z^7 + \ldots \quad \text{with } R = 1.$$

You can verify that this series gives the value of the function at $z = 1/2$ to 3 decimal places with about 5 terms of the series.

You could also find its Taylor series about any other $z_0 \neq \pm i$ Try $z_0 = \pm 1$.

There are other possible formulae for the coefficients in a Taylor series (or expansion.) From the generalized Cauchy formula one has

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

where C is a simple closed contour around z_0 with $f(z)$ analytic inside C. Any choice of C is ine so usually it is just a circle.

Isolated Singularities

Many of the functions that occur here are analytic in a domain $D \setminus \{z_0\}$ where z_0 is "surrounded" by D. In this case z_0 is called a **singularity** of the analytic function f on D. the typical examples are

$$f(z) \;:=\; \frac{c}{(z-z_0)^k}, \quad k \in \mathbb{N}$$

Another example would be a function such as $f(z) \;:=\; exp(-1/z^2)$ for $z \neq 0$. From the series for the exponential function, this is given formally by

$$f(z) \;=\; 1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} - \frac{1}{z^8} + \ldots$$

Suppose z_0 is an isolated singularity of an analytic function $f(z)$, then
(i) z_0 is a **pole of order k** for $f(z)$ provided there is a nonzero complex number b_k and a $k \in \mathbb{N}$ such that

$$\lim_{z \to z_0} (z - z_0)^k f(z) = b_k.$$

(ii) z_0 is an **essential singularity** of $f(z)$ when it is a singular point but the above limit does not exist for any positive integer k.

The function $f(z) := exp(-1/z^2)$ for $z \neq 0$ has an essential singularity at the origin. Sometimes we say a function has a removable singularity if it is defined by a formula that appears to make $f(z_0)$ bad, but really the $\lim_{z \to z_0} (z - z_0) f(z) = b$ is finite.

Laurent Series

Suppose $f(z)$ is an analytic function on an annular domain $A := \{z : r_1 < |z - z_0| < r_2\}$ (If $r_1 = 0$ this is called a deleted disk; when $r_1 > 0$ this is an annulus.)

In this case the function can be written as a power series in both $(z - z_0)$ and $(z - z_0)^{-1}$ so that

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \quad \text{for } z \in A$$

This series converges everywhere in A and the coefficients are given by

$$c_k := \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

where k is any positive or negative integer and C is any **simple closed contour in A that goes around z_0.**