Complex Numbers and Functions

So far this class has treated the basic algebra and definitions needed to work with complex numbers and functions. You should understand and be able to use

1. Complex numbers and complex arithmetic, polar form, \(|z|\), arg \(z\).
2. The definitions of various types of subsets of \(\mathbb{C}\), (open, closed, connected, bounded, boundary domain and region).
3. Limits and convergence of a sequence, continuity of a function, rules for sums, products, quotients and compositions.
5. Complex Polynomials, Exponential, Trigonometric and Hyperbolic functions.
In particular we have used derivatives of functions of a complex variable to show that the real and imaginary parts of a differentiable function are related by the Cauchy-Riemann equations. Also studied the exponential function and showed that for complex $z$ - the exponential, trigonometric and hyperbolic functions all are related.

Most of this theory is quite similar to the theory of 2-dimensional calculus. In two dimensional calculus, one can define 2d integration over subsets of the plane and also line integrals along “curves” or ”contours”. These are piecewise differentiable curves in the plane and very often piecewise linear (sawtooth) curves are used. So complex integration usually refers to results about the line integrals of complex functions along curves in the complex plane. This is chapter 4 in Saff and Snider and usually the next topic in most next texts on complex functions.
First we need the definitions for complex integration. Then these definitions will be used to obtain formulae and results that are used throughout science and engineering. Using complex integrals, many new analytic functions such as Bessel functions, Legendre functions, rational functions, gamma and zeta functions are defined and their properties established.

In the Digital Library of Mathematical Functions at dlmf.nist.gov/4.2 a high proportion of functions and transforms are defined by complex integrals. It is usually easier to prove results about the integrals than about any function defined as a particular solution of a differential equation. The gamma function which is a complex function that satisfies $\Gamma(m + 1) = m!$ for $m \in \mathbb{N}$ is defined by a line integral - but is not a solution of any linear differential equation.
Integrals of complex functions

When \( f(t) \) is a complex valued function of a real variable with \( f(t) = u(t) + iv(t) \), then the integral

\[
\int_a^b f(t) dt := \int_a^b u(t) dt + i \int_a^b v(t) dt
\]

is a complex number whose real and complex parts are the integrals you had in Calculus I.

They have the same properties as ordinary integrals with, in particular,

\[
\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt
\]
Complex Integration

Most of the main theorems are results about contour integrals around “loops” which are piecewise differentiable contours that do not have any endpoints (or boundary). Usually they are part of the boundary of a domain. We will describe properties of these integrals over general contours and then specialize to ”loops”.

The main results in this chapter are Cauchy’s integral theorem which says that a lot of complex (line) integrals around loops are zero and Cauchy’s integral formula which says that $f(z_0)$ is equal to a special integral of $f(z)$ around $z_0$.

This leads to many theorems about analytic functions - including useful formulae and inequalities - and the fundamental theorem of algebra.
The integrals are primarily **contour integrals** of the form

\[ \int_{\Gamma} f(z) \, dz \]

where \( \Gamma \) is a **piecewise smooth curve** in the complex plane and \( f(z) \) is a complex function defined on \( \Gamma \). This integral will be a complex number when \( \Gamma \) has finite length and \( f \) is bounded on \( \Gamma \). How to define this integral?

In elementary integration, you evaluate integrals over an interval \((a, b)\). In 2d calculus there are integrals over domains \(D\) in the plane with respect to area \(dxdy\). Contour integrals are similar to line integrals from calculus 3.
A **curve** (or arc) in the complex plane is the range of a complex function defined on an open or closed interval \( I \subset \mathbb{R} \). That is if \( \zeta : I \rightarrow \mathbb{C} \) is a continuous function then

\[
\Gamma := \{ \zeta(t) : t \in I \}.
\]

When \( I = [t_0, t_1] \) is a closed interval then \( \zeta(t_0), \zeta(t_1) \) are called the **end-points** of the curve. If \( \zeta(t_0) = \zeta(t_1) \), then \( \Gamma \) is said to be a **loop** or a **closed curve**. Occasionally a single point \( z_0 \) is called a loop - defined by the function \( \zeta : [0, 1] \rightarrow \mathbb{C} \) with \( \zeta(t) = z_0 \) for all \( t \).
When a smooth curve is parametrized by a function $\zeta$ then the **arc-length** of the curve can be evaluated. The distance of a point $\zeta(t) = x(t) + iy(t)$ from the initial point $z_0 = \zeta(t_0)$ is

$$s(t) := \int_{t_0}^{t} |\dot{\zeta}(\tau)| \, d\tau.$$ 

This $s(t)$ is called the arc-length of $\zeta(t)$ from $z_0$ along $\Gamma$ and

$$\frac{ds}{dt}(t) = |\dot{\zeta}(t)| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} > 0$$

The length of $\Gamma$ is

$$|\Gamma| = L(\Gamma) := \int_{t_0}^{t_1} |\dot{\zeta}(t)| \, dt.$$
Example: An ellipse center at the origin and with semi-axes $a, b$ has the parametric form

$$\zeta(\theta) := a \cos \theta + i b \sin \theta \quad 0 \leq \theta \leq 2\pi$$

This is a simple loop. The length of the arc of this ellipse between $z_0 = a$ and $z_1 = i b$ corresponds to the arc going from $\theta = 0$ to $\theta = \pi/2$ so has length

$$L(\Gamma) := \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta.$$

The perimeter of the ellipse is 4 times this value and the area inside the ellipse is $\pi a b$. This is an elliptic integral which does not have a simple expression in terms of functions you have seen when $a \neq b$. 
Suppose that we choose a grid $G := \{z_0, z_1, \ldots, z_M\}$ of consecutive points on $\Gamma$ from $z_0 = \zeta(a)$ to $z_M = \zeta(b)$. Then

$$I_G(f) := \sum_{j=1}^{M-1} f(\zeta_j) (z_{j+1} - z_j) \quad \text{with} \quad \zeta_j \in (z_j, z_{j+1})$$

is a Riemann sum for this integral contour integral. Let $h_G := \max_j |z_{j+1} - z_j|$ be the grid mesh. Then

$$I_\Gamma(f) := \int_\Gamma f(z) \, dz := \lim_{h_G \to 0} \sum_{j=0}^{M-1} f(\zeta_j) (z_{j+1} - z_j)$$

This limit should be a complex number.
This contour integral depends on both the contour $\Gamma$ and the function $f(z)$. When $\Gamma$ is a $C^1$-curve in the complex plane given by $\Gamma := \{z(t) : t_0 \leq t \leq t_1\}$, then a careful verification will show that

$$I_\Gamma(f) := \int_{\Gamma} f(z) \, dz = \int_{t_0}^{t_1} f(z(t)) \dot{z}(t) \, dt$$

This last integral is just an ordinary 1-d Riemann integral with an integrand that is a continuous, complex valued function of $t$. The integrand has the form $F(t) := f(z(t)) \dot{z}(t)$ so the integral is evaluated by finding the values of $\int_{t_0}^{t_1} \text{Re}(F(t)) \, dt$ and $\int_{t_0}^{t_1} \text{Im}(F(t)) \, dt$ where $\text{Re}(F(t))$ and $\text{Im}(F(t))$ are the real and imaginary parts of $F(t)$. 
When $f, g$ are continuous complex functions defined on a directed smooth curve (contour) $\Gamma$ of finite length, the contour integral $\int_\Gamma (f)$ will be a complex number. They obey the usual rules including Linearity

$$\int_\Gamma [c_1 f_1(z) + c_2 f_2(z)] \, dz = \int_\Gamma c_1 f_1(z) \, dz + \int_\Gamma c_2 f_2(z) \, dz$$

Triangle inequality: If $|f(z)| \leq M$ on $\Gamma$, then

$$\left| \int_\Gamma f(z) \, dz \right| \leq M \, L(\Gamma).$$

Orientation: Let $-\Gamma$ be the curve in the reverse direction from $\zeta(b)$ to $\zeta(a)$. Then

$$\int_{-\Gamma} f(z) \, dz = - \int_\Gamma f(z) \, dz$$
Example: Let $z_0 \in \mathbb{C}$ and $C$ be the circle of center $z_0$ and radius $r$. Find the integrals $\int_C (z - z_0)^n \, dz$ with $n \in \mathbb{Z}$.

First choose a parametrization of $C$ such as
\[ \zeta(\theta) := z_0 + r \, e^{i\theta} \text{ for } 0 \leq \theta \leq 2\pi. \]
This is a regular arc and a loop. On $C$, $f(\zeta(\theta)) = r^n \, e^{in\theta}$. Thus
\[
\int_C (z - z_0)^n \, dz = \int_0^{2\pi} ir^{n+1} \, e^{i(n+1)\theta} \, d\theta
\]

When $n \neq -1$, this integral with respect to $\theta$ is zero. When $n = -1$ one finds that
\[
\int_C \frac{1}{z - z_0} \, dz = 2\pi i
\]
so this integral is independent of $r$!
Very often we wish to work with curves that are not smooth curves - but have "corners" or "kinks". For example triangles, rectangles, polygons or similar sets.

A **contour** is a curve in the complex plane that consists of a finite number of $C^1$—adjacent segments $\gamma_1, \ldots, \gamma_n$ with end-points (=complex numbers) $\{z_j : 0 \leq j \leq n\}$ That is the initial point of the component $\gamma_{j+1}$ is the final point of the component $\gamma_j$ for $1 \leq j \leq n - 1$, and we can write

$$\Gamma = \bigcup_{j=1}^{n} \gamma_j$$

The usual examples are piecewise straightlines such as parts of the edge of a polygon. In particular a contour has an orientation, or direction, with $-\Gamma$ is the same set but with the reverse ordering. All the preceding rules for evaluating integrals hold when contours are used in place of arcs.
Then the contour integral of a function along a general contour $\Gamma$ is the sum of the integrals of the function along each segment

$$l_\Gamma(f) := \int_\Gamma f(z) \, dz = \sum_{j=1}^n \int_{\gamma_j} f(z) \, dz$$

We will prove some results about how these integrals depend on specific choices of contours. We often choose piecewise linear or piecewise circular arcs because they are usually easier to evaluate since the function $\dot{\zeta}$ is simpler. In the following we will often subdivide contours into a finite number of simpler segments.
A curve $\Gamma$ is self-intersecting if there are distinct values $t_1, t_2$ in $I$ that are not both end points and such that $\zeta(t_1) = \zeta(t_2)$. If a curve is not self-intersecting then the function $\zeta$ is 1-1 - except possibly at the end-points if it is a loop. Often a finite length non-self-intersecting loop is called a **simple loop**.

A curve $\Gamma$ is said to be a **directed smooth curve (or arc)** when it is defined by a function $\zeta : [a, b] \rightarrow \mathbb{C}$ and

(i) $\Gamma$ is not self-intersecting,
(ii) $\dot{\zeta}(t)$ is non-zero, finite and continuous on $I := [a,b]$.

Here $\dot{\zeta}(t) := \dot{x}(t) + i\dot{y}(t)$ is the usual (1-d) derivative with respect to $t$. In particular a curve has a direction; it goes from $\zeta(t_0)$ to $\zeta(t_1)$. $\zeta$ provides a **parametrization** of $\Gamma$. The same curve can have different parametrizations.
Example. Evaluate the integral of \( f(z) = z \) along the segment (directed smooth arc) \( \Gamma \) of an ellipse of semiaxes 2, 1 in the first quadrant.

This segment may be described parametrically by \( z(t) := x(t) + iy(t) \) with \( x(t) = 2 \cos t, \ y(t) = \sin t \) with \( 0 \leq t \leq \pi/2 \). Then

\[
\int_{\Gamma} z \, dz = \int_{0}^{\pi/2} (2 \cos t + i \sin t)(-2 \sin t + i \cos t) \, dt
\]

The real part of this integral

\[
= - \int_{0}^{\pi/2} 5 \sin t \cos t \, dt = -\frac{5}{2}.
\]

Suggest that you evaluate the imaginary part of this number using the formulae you know.
Complex Fundamental Theorem of Integration

Suppose \( D \) is a domain and \( f : D \to \mathbb{C} \) is a differentiable function on \( D \) with the anti-derivative \( F \) on \( D \). That is \( F'(z) = f(z) \) on \( D \). If \( \Gamma \) is a contour in \( D \) consisting of a finite number of directed smooth curves \( \gamma_j \) with end points \( \{z_j : 0 \leq j \leq n\} \), then

\[
\int_{\Gamma} f(z) \, dz = F(z_n) - F(z_0)
\]

Note that if the direction of \( \Gamma \) is reversed, the sign of this integral is changed!

This is a complex form of a fundamental theorem of calculus for anti-derivatives of 1-d functions. It says that as long as you know an antiderivative on a domain, then the value of the integral depends only on the end points of a contour in that domain.
For the integral along a quarter of an ellipse one could use the fact that $F(z) = z^2/2$ is an antiderivative of $f(z) = z$. The end points are $z_0 = 2$ and $z_1 = i$. Then the Complex Fundamental theorem says

$$\int_{\Gamma} z \, dz = F(i) - F(2) = -(1 + 4)/2 = -5/2.$$ 

Did you find the complex part correctly above?

Above we showed that the integral of a power function like $(z - \zeta)^m$ is zero around any circle centered at $\zeta$ when $m \neq -1$. If you calculate many explicit such integrals you will find that this holds for a large class of functions $f(z)$ and most piecewise $C^1$–loops.
When the function $f(z)$ is complex valued and differentiable in a domain $D$ then the result holds whenever $\Gamma$ is a directed piecewise smooth curve that also is a loop - so it has no end points and is non-self intersecting.

**Cauchy -Goursat Theorem**  
Suppose a function $f(z)$ is differentiable on a domain $D$ and $C$ is a simple smooth loop in then

$$\int_{C} f(z) \, dz = 0$$

Note that this theorem doesn’t require that we know an antiderivative of $f(z)$ on $D$. So it often used for evaluating integrals when we don’t know an antiderivative.
Suppose that a function $f(z)$ is analytic in a region $D$, $z_0, z_1$ are two points in $D$ and $C_1, C_2$ are two curves from $z_0$ to $z_1$ that lie in $D$ and do not intersect. Then the curve obtained by joining $-C_2$ to $C_1$ (or $-C_1$ to $C_2$) is a simple loop in $D$. So

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz$$

That is the value of a contour integral along two different contours joining from a point $z_0$ to another point $z_1$ in $D$ does not depend on the contour you choose provided there are no self-intersections and the contour stays in $D$. 
Another way of saying this is that when \( f(z) \) is analytic in a region \( D \), \( z_0, z \) are points in the region, then the integral

\[
F(z) := \int_{z_0}^{z} f(z) \, dz = \int_{C} f(z) \, dz
\]

for any curve \( C \) that goes from \( z_0 \) to \( z \) and remains in \( D \). That is this indefinite contour integral is independent of the path from \( z_0 \) to \( z \)!

This integral may be defined for all \( z \in D \) so \( F \) is a complex valued function on \( D \). One can then prove that this function \( F(z) \) is differentiable (or analytic) on \( D \) with

\[
F'(z) = \frac{dF}{dz}(z) = f(z) \quad \text{for all} \quad z \in D.
\]

This is another form of the fundamental theorem for (continuous) functions of a complex variable.
Any simple smooth loop can be traversed in one of two directions. The Jordan curve theorem says that such a loop $\Gamma$ divides the complex plane into two domains. A bounded domain $D$ inside the loop and an unbounded domain $\tilde{D}$ outside the loop. Then $\mathbb{C} = D \cup \Gamma \cup \tilde{D}$. Try drawing some loops and check this out - identify the inside and outside domains.

Mathematicians say that a loop is positively oriented if it is traversed so that the bounded domain is on the left hand side as you go along the loop. For simple circles or ellipses with center inside the loop this means the loop is traversed in an anti-clockwise direction. with the angle at the center increasing in $\theta$. However loops may have very complicated behavior so this orientation is best described in terms of what’s inside and outside.
Suppose that \( D \) is a bounded domain in \( \mathbb{C} \), whose boundary consist of simple loops that are arcs of finite length. If the boundary is just one loop, the domain is said to be simply connected. If the boundary consists of \( J+1 \) disjoint loops, then the domain is said to have \( J \) holes and be multiply connected. There will be an outside boundary \( \Gamma_0 \) which is taken to be positively oriented and \( J \) boundaries \( \Gamma_m \) with \( 1 \leq m \leq M \) that are negatively oriented.

When there is just one hole we have an **annular domain** with an outside boundary \( \Gamma_0 \) and an inside boundary \( \Gamma_1 \) and the domain is the set of all point that lie outside \( \Gamma_1 \) but inside \( \Gamma_0 \).
In this class, we will mostly work with either simply connected or annular domains, but all the results have generalizations to multiply connected domains. These formulae require extra rules and ”winding numbers”.

In particular we will often work with functions that are differentiable on a bounded domain, continuous on the closed set $D \cup C$ and require that the boundary $C$ be a simple positively oriented smooth loop.
Cauchy’s Integral Formula

Let $D$ be the simply connected domain inside a simple positively oriented loop $C$. If $f$ is differentiable (analytic) on domain $D$, then the values of $f$ in $D$ are given by a contour integral. Namely

$$ f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz \quad \text{for } z_0 \in D $$

That is, the value of an analytic function at a point $z_0$ inside a loop $C$ can be found by knowing the value of this integral along a contour around the point. When one knows the values of an analytic function on a circle or a rectangle or a general polygon in the complex plane, then this Integral formula gives the value of the function at any point inside the loop.
Example. Evaluate the integral of $g(z) := \cos z/z$ around the circle $C$ of center the origin and radius 1.

This is an integral of the type in Cauchy’s Integral formula with $f(z) = \cos z$, $z_0 = 0$. From Cauchy’s integral formula, one sees that

$$\int_C \frac{\cos z}{z} \, dz = 2\pi i \cos(0) = 2\pi i$$

If you substitute the parametric equation for a circle in this integral you get a very difficult integral to evaluate. Moreover the same integral formula holds when $C$ is any simple loop that has the origin on its inside. So $C$ could be an ellipse or a very complicated non-self-intersecting loop.
Example. Evaluate the integral of $g(z) := \cos z/(3z - \pi)$ around the circle $C$ of center the origin and radius 1.

This is an integral of the type in Cauchy's Integral formula with $f(z) = \cos z/3$, $z_0 = \pi/3$. This function $g(z)$ is differentiable inside this loop as its only singularity occurs at $z_0 = \pi/3$ and $\pi > 3$ so this point is outside $C$. Thus the integral around the circle of radius 1, is zero from the Cauchy-Goursat theorem.
Example. Evaluate the integral of \( g(z) := \frac{\cos z}{3z - \pi} \) around the circle \( C \) of center the origin and radius 1.

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If \( C \) is the a p.o. circle of radius \( R > 1.2 \), then \( z_0 \) is inside the circle and then

\[
\int_C \frac{\cos z}{(3z - \pi)} \ dz = \frac{2i\pi}{3} \cos \left( \frac{\pi}{3} \right) = \frac{i\pi}{3}
\]
Generalized Cauchy’s Integral Formula

Let $D$ be the simply connected domain inside a simple positively oriented loop $C$. If $f$ is analytic on domain $D$, then the values of the $k$-th derivative $f^{(k)}(z)$ at a point $z_0$ in $D$ is given by a contour integral. Namely

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+1}} \, dz \quad \text{for } z_0 \in D$$

Since each of these integrals can be evaluated and is finite, this formula is the basis of the usual proof that if $f(z)$ is analytic inside and on the loop $C$ then all its derivatives exist and are finite at points inside $C$. 
The result that is used to prove the Generalized Cauchy Integral Formula is that if $\Gamma$ is a smooth arc of finite length, and a complex function $g$ is continuous on the arc, then the function

$$G(z) := \int_{\Gamma} \frac{g(\zeta)}{\zeta - z} d\zeta$$

is analytic at any point $z \notin \gamma$. Moreover

$$G'(z) = \frac{dG}{dz}(z) = \int_{\Gamma} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$$

$$G''(z) = \frac{d^2G}{dz^2}(z) = 2 \int_{\Gamma} \frac{g(\zeta)}{(\zeta - z)^3} d\zeta$$

... and all these integrals are complex numbers. This is called "differentiation under the integral sign".
Examples    Evaluate the following integrals around a closed loop $C$ of your choice. Say what loops will make the integrals zero - or not.

\[ \int_C \frac{\sin z}{z} \, dz \quad \int_C \frac{\sin z}{2z - \pi} \, dz \]

\[ \int_C \frac{z^3 + z - 1}{(2z - 4)^2} \, dz \quad \int_C \frac{e^{-2z}}{z^4} \, dz \]

When you see integrals of fractions where there are polynomials or other simple functions in the denominator, look to see where this denominator is zero. These are the "singularities" and integrals on loops around the singular points may be non zero.
The formula about the derivatives of contour integrals holds whenever $\Gamma$ is a smooth arc and thus it holds for contours. and then for loops. So we can apply it to the expression in the Cauchy integral formula for any differentiable function $f(z)$ on a simply connected domain. Thus the derivative

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} \, dz \quad \text{for } z_0 \in D$$

and also

$$\frac{d^2F}{dz^2}(z_0) := f''(z_0) = \frac{1}{\pi i} \int_C \frac{f(z)}{(z-z_0)^3} \, dz \quad \text{for } z_0 \in D$$

and the formula for higher derivatives follows by induction. That is if a function of a complex variable has one derivative in a domain then it has any derivative that you may want to evaluate. This is not true for real valued functions in calculus 1 and 2.
Bounds on Analytic Functions and Derivatives.

The importance of the Cauchy integral formulae is that when we know an analytic function on a contour \( \Gamma \) then you can evaluate it at any point inside the contour - by evaluating a contour integral. This allows one to also find bounds on the values of the function.

Suppose that a function is analytic on the circle \( C_R \) center \( z_0 \) and radius \( R \). Suppose that \( |f(\zeta)| \leq M \) for \( \zeta \in C_R \). Then \( |f(z)| \leq M \) inside \( C_R \) and Cauchy’s integral formula says that

\[
|f'(z_0)| \leq \frac{M}{R}, \quad |f''(z_0)| \leq \frac{2M}{R^2}, \quad \text{and}
\]

\[
|f^{(k)}(z_0)| \leq \frac{M \, k!}{R^k}.
\]
A function $f$ on $\mathbb{C}$ is said to be entire provided it is analytic for all $z$.

**Theorem (Liouville)** If an entire function is bounded on $\mathbb{C}$, then it is constant on $\mathbb{C}$.

Suppose now that $p(z)$ is a polynomial of degree $n$. That is

$$p(z) := a_0z^n + a_1z^{n-1} + \ldots + a_{n-1}z + a_n$$

with $a_0 \neq 0$ and each $a_j \in \mathbb{C}$. Then

$$\lim_{|z| \to \infty} \frac{p(z)}{z^n} = a_0.$$

A basic question in algebra has been how do you find the zeros of $p(z)$. That is to find the solutions of $p(z) = 0$.

Are there formulae for these zeroes? Yes for $n = 2, 3, 4$. No for $n \geq 5$. 