Introduction to Complex Analysis
Math 3364 Fall 2020

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The syllabus and course description are available on my web site and also at AccessUH. They describe how the course will be graded and when the midterm exam is. The final exam is as in the UH website.

This course will not follow any current text very closely. There are many texts available and they all cover the material in a similar way for the first few chapters. Nearly all the results to be described in this course were known by 1900 and many of the texts are quite similar. The most recent texts often include computational and graphical treatments of many topics - which shall not be covered here.

In the beginning there are the **integers**

$$\mathbb{N} := \{1, 2, 3, 4, \ldots, n, \ldots\} \tag{1}$$

The set of **integers** and the set of **rational numbers** are

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots, \pm n. \ldots\} \tag{2}$$

$$\mathbb{Q} := \{\pm m/n : m \in \mathbb{Z}, \quad n \in \mathbb{N}\} \tag{3}$$

The set of **real numbers** is $\mathbb{R} := (-\infty, \infty)$ The set of **extended real numbers** is $\overline{\mathbb{R}} := [-\infty, \infty]$

$\pm\infty$ can be defined and are numbers (but not real numbers)

Complex numbers and functions are used throughout science and engineering - despite the fact that they are often called imaginary numbers.

A **complex number** is an expression of the form $x + iy$ where x, y are real numbers and i is a symbol to identify the number as being complex. Usually complex numbers are represented by w, z or a Greek lower case letter. When $z = x + iy$ then x = Re(z) the is called the **real part of z**, y = Im(z) is the **imaginary part of z**.

Two complex numbers are said to be equal if their real and imaginary parts are the same.

Algebra of Complex Numbers

The set of all complex numbers is denoted $\mathbb{C}$ and the complex numbers 0 and 1 are $0 = 0 + i\,0$ and $1 = 1 + i\,0$ Just as for real numbers, there are two basic operations; addition and multiplication defined by

Addition: $z_1 + z_2 \;:=\; (x_1 + x_2) + i\,(y_1 + y_2)$

Multiplication: $z_1.z_2 \;:=\; (x_1x_2 - y_1y_2) \;+\; i\,(x_2y_1 + x_1y_2)$

Note that addition is standard vector addition in $\mathbb{R}^2$ while multiplication is usual multiplication with the convention that $i^2 = -1$. It is different to the inner (or dot) product of vectors.

The inverse operations of subtraction and division that $-z = -x - iy$ and $w = z^{-1}$ are the complex numbers that satisfy

$$z \;+\; (-z) \;=\; 0 \quad \text{and} \quad z \cdot w \;=\; 1$$

$\mathbb{C}$ is a vector space and a commutative algebra with respect to the operations $+, .$ That is, both addition and multiplication obey the commutative and associative rules. Addition and multiplication satisfy the usual distributive rules of real numbers - so all your real algebra rules carry over to complex numbers and functions..

When $z = x + iy$ is a complex number then the **complex conjugate** of z is $\bar{z} := x - iy$.

The **modulus** of z is $|z| := \sqrt{x^2 + y^2}$ so $|z|^2 = z.\bar{z} = x^2 + y^2$. When $z \neq 0$ then the **multiplicative inverse** of z is the complex number w such that $w.z = z.w = 1$. Thus

$$z^{-1} := \frac{1}{z} := \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

Complex numbers are often regarded as points in the plane with Cartesian coordinates (x, y) so $\mathbb{C}$ is **isomorphic** to the plane $\mathbb{R}^2$. When $z = x + iy$ is a complex number then the **complex conjugate** of z is $\bar{z} := x - iy$. This is the reflection of a complex number z about the x-axis.

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There are many simple rules for these operations - section 1.2, page 5 of the text. The rules for conjugation and moduli also are quite simple. Complex numbers obey all the same algebraic propeties as real numbers - and this algebraic structure is called a **field** by mathematicians.

Exercise: For each of the following numbers z, evaluate $\bar{z}, |\, z \,|, 1/z$ and also evaluate various products.

$$1+i, \quad 1-i, \quad 3+4i, \quad \pm 1 + i\sqrt{3}, \quad \pm 5 + i\sqrt{12}$$

Do this witout calculators or computational assistance and then use some program to check your answers.

Most mathematical calculators and software packages have routines for complex arithmetic and I suggest that each of you learn to use such a calculator so you can check answers to homework problems.

Question: Do you have software on your computer that does complex arithmetic? Matlab, Mathematica, Maple, ...? Possibly Excel or other spreadsheet software ?

Strongly recommend that you find, and learn how to use, some software that does complex arithmetic.

Geometry of Complex Plane

When z is a **nonzero** complex number, then the **argument** of z is the angle θ where

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}. \tag{4}$$

There is a unique value of theta in either $[0, 2\pi)$ or $(-\pi, \pi]$ such that (4) holds. (Radians are the units for angles here). From this equation you see that

$$z = |z|\,(\cos\theta + i\,\sin\theta)$$

and write $arg(z) := \theta$. This is called the **polar form** of the complex number z and represents z in terms of a length and an angle. Occasionally also "$cis\theta$" for $\cos\theta + i\,\sin\theta$.

The **triangle inequality** is the result that

$$|z_1 \ + \ z_2| \ \leq \ |z_1| \ + \ |z_2| \quad \text{for all} \ \ z_1, z_2 \in \mathbb{C}.$$

It is a nice exercise in geometry to prove this.

The polar form is most useful for multiplication. When z_1, z_2 are **nonzero** complex numbers with arguments θ_1, θ_2 then

$$z_1 \, . \, z_2 \ = \ |z_1| \, . \, |z_2| \ cis(\theta_1 + \theta_2)$$

This has a geometrical interpretation as $|z_1.z_2| = |z_1|.|z_2|$ and

$$arg(z_1 \, . \, z_2) \ = \ arg(z_1) + arg(z_2)$$

$$arg\left(\frac{z_1}{z_2}\right) \ = \ arg(z_1) \ - \ arg(z_2)$$

There are lots of amazing formulae involving complex numbers and functions - and they may be used in many different ways. Many geometrical questions also can be viewed using complex geometry so we will look at some of them.

The Exponential function

The ordinary exponential function is the function $f(x) := e^x$. It is the unique solution of the equation

$$\frac{df}{dx} = f(x) \qquad f(0) = 1$$

and has the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \frac{x^m}{m!} + \ldots .$$

This series converges for every real $x \in \mathbb{R}$ and the function has the property $e^{x_1 + x_2} = e^{x_1} . e^{x_2}$

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Note that if a is a positive real number the general exponential function is $g(x) = a^x$ and has $g'(x) = \ln a . a^x$.

The Complex Exponential function

All the preceding formulae still hold if x is replaced by the complex number $z \in \mathbb{C}$. In particular define

$$e^z \;:=\; e^x \,.\, e^{yi} \quad \text{with}$$

$$e^{yi} \;=\; 1 \;+\; yi \;-\; \frac{y^2}{2!} \;+\; \ldots \;+\; \frac{(iy)^m}{m!} \;+\; \ldots .$$

From the Taylor series for sin and cosine functions, this becomes

$$e^z \;:=\; e^{x+\,iy} \;=\; e^x\,(\cos y \;+\; i \sin y).$$

Note that $e^0 = 1$ so $e^{0i} = 1$ when $y = 0$ so this complex definition agrees with the definition that was given in Calc I.

Thus $e^{i\theta} \; = \; \cos\theta \; + \; i \; \sin\theta$ and the **polar representation** of complex numbers

$$z \;=\; |z| \; e^{i\theta} \;=\; r \; e^{i\theta}$$

Also one finds the formulae

$$\cos\theta \;=\; \frac{e^{i\theta} \;+\; e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta \;=\; \frac{e^{i\theta} \;-\; e^{-i\theta}}{2i}$$

When $\quad z_1 = r_1\, e^{i\theta_1}, \; z_2 = r_2\, e^{i\theta_2} \quad$ then

$$z_1 . z_2 \;=\; r_1\, r_2\, e^{i(\theta_1+\theta_2)} \quad \text{and} \quad \frac{z_1}{z_2} \;=\; \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$$

provided $r_2 \neq 0$.

The preceding results lead to **DeMoivre's theorem**

$$\cos m\theta + i \sin m\theta \; = \; e^{im\theta} \; = \; (\cos\theta + i\sin\theta)^m$$

These enable the derivation of formulae for cos of $m\theta$ in terms of cos of θ

$$\cos m\theta = T_m(\cos\theta)$$

T_m is the m-th Chebychev polynomial and is a polynomial of degree m. Unfortunately $\sin m\theta$ is not a polynomial in $\sin\theta$. (Check when m =2,3.)

Example. The quartic expansion is
$(a+b)^4 \; = \; a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$. Then
$\cos 4\theta \; = \; (\cos\theta)^4 - 6(\cos\theta)^2(\sin\theta)^2 + (\sin\theta)^4$
by equating real and imaginary parts. Thus
$\cos 4\theta \; = \; 8(\cos\theta)^4 - 8(\cos\theta)^2 + 1.$

Euler's formula has $x = 0, y = \theta$ in the definition of e^z, so

$$e^{i\theta} \;=\; \cos\theta \;+\; i\;\sin\theta$$

See Wikipedia for more about the history. When θ is $\pi/2,\;\pi, 3\pi/2$ respectively we have

$$e^{i\pi/2} \;=\; i, \qquad e^{i\pi} \;=\; -1, \qquad e^{3i\pi/2} \;=\; -i$$

These are Euler's identities and

$$e^{i(\theta+2k\pi)} \;=\; e^{i\theta} \qquad \text{for all} \;\; k \in \mathbb{Z}$$

That is $e^{i\theta}$ is periodic of period 2π. A function is periodic of period T provided $f(t+T) = f(t)$ for all $t \in \mathbb{R}$ or $f(t+kT) \;=\; f(t)$ for all t and $k \in \mathbb{Z}$.

Roots of Unity

Problem: Find the solutions of $z^m = 1$ when $m \in \mathbb{N}$.
Solutions of this equation are called the **m-th roots of unity.**

When m =2, the solutions of $z^2 = 1$ are $z = \pm 1$ since

$$z^2 \ - \ 1 \ = \ (z-1).(z+1).$$

When $m = 3$ one has $z^3 - 1 \ = \ (z-1).(z^2+z+1)$.
For any $m \geq 2$, one has the geometric series

$$z^m - 1 \ = \ (z-1).(1+z+z^2+\ldots+z^{m-1})$$

so $z = 1$ is always a solution and $z = -1$ is also a solution when m is even. These are the only possible real solutions.

From the Euler identities $e^{2ki\pi} = 1$ for any integer k. Thus $\omega_m := \exp 2i\pi/m$ satisfies $\omega_m^m = 1$.

Similarly ω_m^k satisfies the equation for any integer k. However only the points where $k = 0, 1, 2, \ldots, m-1$ are different - so there are m distinct complex m-th roots of unity. Thus when $m \geq 3$ there always are complex roots as well as the real roots.

Example. the solutions of $z^4 - 1 = 0$ are $z = \pm 1, \pm i$ as $(z^4 - 1) = (z^2 - 1).(z^2 + 1)$.

In general there will be m distinct solutions of $z^m = a$ when a is a non-zero complex number.

Suppose $a = |a|e^{i\theta}$. Then the solutions are

$$z_k := |a|^{1/m} \, e^{i(\theta+2k\pi)/m} \qquad \text{for } k = 0, 1, \ldots, m-1$$

The solutions of the quadratic equation

$$az^2 + bz + c = 0 \quad \text{with } a \neq 0 \quad \text{are}$$

$$z_\pm := \frac{1}{2a} \left[-b \pm [b^2 - 4ac]^{1/2} \right]$$

where the last term is taken to be a complex square root.

If a polynomial $p(z) = z^m + a_1 z^{m-1} + \ldots + a_{m-1} z + a_m$ can be factored as the product of other polynomials $p(z) = p_1(z).p_2(z)$ then the solutions of $p(z) = 0$ are either solutions of $p_1(z) = 0$ or of $p_2(z) = 0$.

Sets of Complex Numbers

The **open disk center** z_0 and radius ρ is the set

$$B_\rho(z_0) \; := \; \{z \in \mathbb{C} : |z - z_0| < \rho\}$$

This set is often called the ρ neighborhood of z_0. A set S is said to be **bounded** if there is an R such that $S \subset B_R(0)$.

When S is a set of complex numbers then z_0 is an **interior point** of S provided there is a $\rho > 0$ such that $B_\rho(z_0) \subset S$.

S is said to be an open set if for each point of S is an interior point.

The **closed disk center** z_0 and radius ρ is the set

$$\overline{B}_\rho(z_0) \; := \; \{z \in \mathbb{C} : |z - z_0| \leq \rho\}$$

When z_0, z_1 are complex numbers then the **interval** $[z_0, z_1]$ is the straight line joining these endpoints. That is,

$$[z_0, z_1] \ := \ \{(1-t)z_0 + tz_1 : 0 \leq t \leq 1\}$$

A **polygonal path** with nodes $z_0, z_1, z_2, \ldots, z_m$ is the union of the straight lines joining these points (in order).

$$\text{Path} \ := \ [z_0, z_1] \cup [z_1, z_2] \cup \ldots [z_{m-1}, z_m]$$

A (non-empty) subset S of $\mathbb{C}$ is said to be **connected** if any two points w, z in S may be joined by a polygonal path that is a subset of S.

A (non-empty) open connected subset S of $\mathbb{C}$ is called a **domain**.

The complement of a set S is the set $\tilde{S} := \mathbb{C} \backslash S$ of all complex numbers that are not in S.

A point $z_0 \in \mathbb{C}$ lies in the boundary of a (nonempty) set S provided every open disk centered at z_0 contains points in S and also points in $\tilde{S}$.

The **boundary of S** is the set of all boundary points of S and is denoted ∂S. The set S is said to be closed if it contains all of its boundary points.

A set that is bounded and closed is said to be **compact**. A set that is not bounded is said to be **unbounded**.

Complex Functions

This course is about the theory of **complex valued functions** of a complex variable. We prove results about continuous and differentiable functions and the evaluation of complex integrals.

The simplest such functions are polynomials

$$p(z) \;:=\; z^m + a_1 z^{m-1} + \ldots + a_{m-1} z + a_m$$

with complex coefficients $a_1, a_2, \ldots, a_m$ and the exponential, hyperbolic and trigonometric functions

$$e^z,\; sinh(z),\; cosh(z),\; sin(z),\; cos(z), \ldots$$

Are the formulae you know from real variable calculus still true for complex analysis, what changes ae needed and/or are there new results?

Suppose $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$ is a function. Then S is called the **domain** of the function f and the set $f(S) := \{f(z) : z \in S\}$ is called the **range** of f.

Write $z = x + iy$, then the complex function has a real representation

$$f(z) = u(x,y) + i\,v(x,y) \quad \text{for } z \in S$$

Here u, v are called the real and imaginary parts of f.

Example: Find the real and imaginary parts of $f(z) = z^3$ or $f(z) := z^2 + iz - e^{i\theta}$ or $f(z) = 1/z$.

Usually we will require that the domain S of a function f also be a domain in the sense of being an open connected set. For example the function $f(z) := 1/(z-a)$ has domain $D = \mathbb{C} \backslash \{a\}$ - and this is also a domain in the sense of being an open connected set in $\mathbb{C}$. Note that the value of $f(a) = 1/0$ - which is not a complex number - so a cannot be in the domain of this function.

If D is a topological domain and $z_0 \in D$ then there always is an open disk $B_\delta(z_0) \subset D$ such that $f(z)$ is a complex number for all $z \in B_\delta(z_0)$.

In the textbook (see section 2.2) they state many results starting with *"If f is a function defined in a neighborhood of* z_0*"* or *" If* $f(z)$ *is continuous at* z_0*".* When $f : D \to \mathbb{C}$ is a function and D is a (topological) domain, then f is defined in a neighborhood of each $z_0 \in D$.

Convergence and Continuity

A sequence $\{z_m : m \in \mathbb{N}\}$ of complex numbers **converges to** a complex number ζ as $m \to \infty$ provided for any $\epsilon > 0$ there is an M such that

$$m > M \text{ implies that } |z_m - \zeta| < \epsilon$$

That is $z_m \in B_\epsilon(\zeta)$ for $m > M$.

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That is $z_m \in B_\epsilon(\zeta)$ for $m > M$.

A function $f : S \to \mathbb{C}$ is **continuous at a point** $\zeta \in S$ provided that when a sequence of points $\{z_m\} \subset S$ converges to ζ then $f(z_m)$ converges to $f(\zeta)$. f is **continuous on S** if it is continuous at every point ζ in S.

We write $\lim_{m\to\infty} z_m = \zeta$, and $\lim_{m\to\infty} f(z_m) = f(\zeta)$, respectively.

An alternative definition of continuity at ζ is that for any $\epsilon > 0$, there is a δ such that

$$z \in S \ \& \ |z - \zeta| < \delta \implies |f(z) - f(\zeta)| < \epsilon$$

Properties of limits.

Suppose that $\lim_{z\to\zeta} f(z) = c_1$ and $\lim_{z\to\zeta} g(z) = c_2$, then

(i) $\lim_{z\to\zeta} f(z) \pm g(z) = c_1 \pm c_2$

(ii) $\lim_{z\to\zeta} f(z).g(z) = c_1.c_2$

(iii) $\lim_{z\to\zeta} f(z)/g(z) = c_1/c_2$ when $c_2 \neq 0$.

If f, g are complex functions on a set S that are continuous at ζ then so also are $(f \pm g)(z)$, $f(z).g(z)$. If $g(\zeta) \neq 0$ then also $f(z)/g(z)$ is continuous at ζ.

If g is continuous at ζ and f is continuous at $g(\zeta)$ then $f(g(z))$ is continuous at ζ. These results are proved using limits in the same way as the corresponding results in calculus.

Definition of Derivative

Let D be a domain with $\zeta \in D$ and $f : D \to \mathbb{C}$ be a complex function. The **derivative of f at** ζ is

$$\frac{df}{dz}(\zeta) \;=\; f'(\zeta) \;=\; \lim_{z\to\zeta} \frac{f(z) - f(\zeta)}{z - \zeta}.$$

- whenever (or provided) this limit exists.

The function f is **differentiable on a domain** D provided it is differentiable at every point in D. A function that is differentiable on a domain is said to be **analytic on D**. (Some other adjectives are used in advanced texts.)

A function that is differentiable at $z_0 \in D$ is continuous at z_0.

The usual rules for the derivatives of standard functions hold. For example

$$\frac{d}{dz}\, z^m = m\ z^{m-1} \quad \text{for } m \in \mathbb{N}$$

$$\frac{d}{dz}\, e^{cz} = c\, e^{cz} \quad \text{for } c \in \mathbb{C}$$

$$\frac{d}{dz}\, \sin z = \cos z \qquad \text{and} \qquad \frac{d}{dz}\, \cos z = -\sin z$$

The rules for differentiation of calculus also hold. Namely if f, g are functions defined on a neighborhood of $z \in \mathbb{C}$ then

$$(f \pm g)'(z) = f'(z) \pm g'(z)$$

$$(f\,g)'(z) = f(z)\,g'(z) + f'(z)\,g(z)$$

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2} \quad \text{if} \quad g(z) \neq 0.$$

If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then the chain rule holds

$$\frac{d}{dz} f(g(z_0)) = f'(g(z_0))\,g'(z_0)$$

Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ where u, v are the real and imaginary parts of f.

Theorem (CR) Suppose $f : D \to \mathbb{C}$ is differentiable at $z_0 := x_0 + iy_0$, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

This results is mostly used to verify that a given pair u, v are the real and complex parts of a differentiable function f on a region D. For example $u(x, y) + iv(x, y) = \overline{z} := x - iy$ is not a differentiable function of z. In particular $\overline{z}$ is not a function of the variable z!

Since the Cauchy-Riemann equations are central results for complex analysis I'll prove them now. This is done by considering the limit in the definition of the derivative by taking limits along the x-axis and the y-axis. If the limit exists it must be have same value no matter what direction we approach the point
$z_0 = x_0 + iy_0$.

First take the limit as $t \to 0$ along the line $z(t) = (x_0 + t) \ + \ i\, y_0$, so

$$\frac{f(z(t) - f(z_0)}{z(t) - z_0} \ =$$

$$= \ \frac{1}{t}\left[u(x_0 + t, y_0) - +iv(x_0 + t, y_0) - u(x_0, y_0) - iv(x_0, y_0)\right]$$

Take limits as $t \to 0$, then

$$f'(z_0) \;=\; \frac{\partial u}{\partial x}(x_0, y_0) \;+\; i\,\frac{\partial v}{\partial x}(x_0, y_0)$$

Similarly if you use the line $z(t) = x_0 + i\,(y_0 + t)$ and take the limits as $t \to 0$, you must divide by it, so

$$f'(z_0) \;=\; \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) \;+\; \frac{\partial v}{\partial y}(x_0, y_0)$$

These two expressions must be equal, so one has

$$\frac{\partial u}{\partial x}(x_0, y_0) \;=\; \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

These are the CR equations.

In the text, theorem 5, section 2.4 states that if u, v are continuously differentiable at (x_0, y_0) and they satisfy the CR equations then f is a differentiable function of z at (x_0, yz).

Theorem Suppose $f : D \to \mathbb{C}$ is differentiable and $f'(z) \equiv 0$ on D. Then $f(z) = c$ is constant on D.

Later in the course we shall show that if f is a complex function that is differentiable on a region D, then every derivative function $f'(z), f''(z), \ldots, f^{(m)}(z)$ is a differentiable function on D.

Thus every real and imaginary part $u(x, y), v(x, y)$ also has every derivative a differentiable function on D.

Laplace's Equations

Suppose D is a region in $\mathbb{C}$ and $f(z) = u(x, y) + iv(x, y)$ where u, v are the real and imaginary parts of f.

Theorem Suppose f is differentiable on D and u, v are the real and imaginary parts of f, then u and v are solutions of Laplace's equation on D.

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } D$$

Examples: 1. $f(z) = z^3$ has $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$.

2. e^z has $u(x, y) = e^x \cos y, \; v(x, y) = e^x \sin y$.

You may check that all these u, v are solutions of Laplace's equation. The Laplacian operator is $\Delta u(x) := \mathrm{div}(\nabla u)(x)$ and can be defined in any dimension $n \geq 1$.

Mappings and Complex Functions

Suppose $f : D \to D_1$ is a complex function that maps a domain D in the complex plane into a region D_1. How to visualize f? Cannot use standard graphs since they need 4 dimensional pictures. Instead complex functions are visualized as "mappings".

That is one has a mapping $F : D \to D_1$ with

$$w = f(z) \qquad \text{or} \qquad F(x,y) = u(x,y) + iv(x,y)$$

So the domain D with coordinates (x,y) is mapped into the set D_1 with coordinates (u,v).

Thus the function $f(z) = z^2$ has $w = z^2 = (x^2 - y^2) + 2ixy$ since

$$u(x,y) = x^2 - y^2 \text{ and } \quad v(x,y) = 2xy$$

Examples 1. the function $f(z) = 1/z$ maps nonzero points inside the unit circle into the outside of the unit circle. Also points outside the unit circle into points inside the unit circle as

$$f(r\ e^{i\theta}) \ = \ \frac{1}{r}\ e^{i\theta}$$

Points $e^{i\theta}$ on the unit circle are mapped to $e^{-i\theta} = \cos\theta - i\sin\theta$. Points in the upper half plane $Im(z) > 0$ are mapped to points in the lower half plane $Im(z) < 0$ with modulis has $|f(z)| = 1/r$.

The exponential function is

$$w \ = \ e^{z} \ = \ e^{x}\cos y \ + \ i\,e^{x}\sin y$$

The text has many pictures of mappings associated with different simple functions.

Properties of e^z.

1. $|e^z| = e^x > 0$ for all $z \in \mathbb{C}$. In particular the equation $e^z = 0$ has no complex solutions.
2. $\frac{d}{dz}e^z = e^z$. Hence $f(z) = e^z$ has

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The complex hyperbolic functions are

$$cosh\, z := \frac{1}{2}[e^z + e^{-z}], \quad sinh\, z := \frac{1}{2}[e^z - e^{-z}]$$

The Trigonometric Functions

The complex function e^z is defined using e^x and the real functions $\cos y$, $\sin y$. The complex sines and cosines now are defined as

$$\cos z \; := \; \frac{1}{2}\,[e^{iz} + e^{-iz}], \qquad \sin z \; := \; \frac{1}{2i}\,[e^{iz} - e^{-iz}]$$

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The decomposition into real and complex parts yield

$$\cos z \;=\; \cos x\,\cosh y \;-\; i\,\sin x\,\sinh y$$

$$\sin z \;=\; \sin x\,\cosh y \;+\; i\,\cos x\,\sinh y$$

Note that these do have the values cos(x), sin(x) when x is real and

$$\cos(iy) \;=\; \cosh y \quad \text{and} \qquad \sin(iy) \;=\; i\sinh y$$

The usual addition and subtraction formulae hold

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

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The solutions of $\sin z = 0$ are precisely $z_k = k\pi$ for $k \in \mathbb{Z}$. The solutions of $\cos z = 0$ are precisely $z_k = (k + 1/2)\pi$ for $k \in \mathbb{Z}$. Also

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

A complex function $f(z)$ is said to be **entire** if it is differentiable at every $z \in \mathbb{C}$. Polynomials, exponential function, sinh, cosh, sin cos are entire functions.

Conjugate Harmonic Functions

You have seen that if $f(z) = u(x,y) + iv(x,y)$ is an analytic function in a domain D then u, v are solutions of Laplace's equations on D and they satisfy the Cauchy Riemann equations;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

When u, v are related by these equations then u, v are said to be **conjugate harmonic functions** on D. Thus in the preceding examples the conjugate function of $u(x,y) = e^x \cos y$ is $v(x,y) = e^x \sin y$.

Sometimes you are given an function $u(x,y)$ that is a solution of Laplace's equation and are asked to find its conjugate. Fot example find the conjugate of $u(x,y) = x^3 - 3xy^2$. To do this you have to "integrate" the partial differential equations

$$\frac{\partial v}{\partial y} = 3(x^2 - y^2) \text{ and } \frac{\partial v}{\partial x} = 6xy$$

The Complex Logarithm Function

Logarithms were invented by John Napier before 1600 as a way to make multiplication easier - since addition is much easier than multiplication of most numbers. Newton invented calculus around 1670 and this led to the definition via calculus

$$\ln x \;=\; \log_e x \;:=\; \int_1^x \frac{dt}{t} \quad \text{for } x > 0.$$

In calculus you learn that ln and the exponential function are inverse.

$$y \;=\; \ln x \quad \Longleftrightarrow \quad e^y = x.$$

Our textbook **defines** the complex logarithm by

$$w \;:=\; \log z \quad \text{provided} \quad e^w \;=\; z \qquad (log)$$

and $z \neq 0$. The equation (log) has infinitely many solutions since, whenever w is a solution, so also is $w + 2\pi i$.

Suppose w is a solution of (L) so that

$$e^w \;=\; z \;=\; r\,e^{i\varphi} \qquad (*)$$

Then $|e^w| \;=\; r > 0$, so $Re(w) \;=\; \ln r \;=\; \ln|z|$ and $w = \ln r + i\psi$ implies that ψ satisfies

$$e^{i\psi} \;=\; e^{i\varphi}$$

That is $\psi = \varphi + 2k\pi i$ for any integer k in $\mathbb{Z}$ and

$$w \;:=\; \log z \;:=\; \ln|z| \;+\; i(\varphi + 2k\pi)$$

is a "function" with infinitely many distinct values. The principal value of this function is denoted Log z and is the particular right hand side (RHS) with argument $\varphi \in (-\pi, \pi]$.

The preceding definition allows you to evaluate the logarithm of any non-zero number - even negative ones. Note that if $z = x$ is a positive real number then $Log\, x \; = \; \ln x$.

If z is a negative real number then definition says that $Log\, z \; = \; \ln|z| + \pi i$. If $z = yi$ is pure imaginary, then $Log(yi) \; = \; \ln|y| + \frac{\pi}{2}\, i$ when $y \geq 0$. When $y < 0$ then

$$Log(yi) \; = \; \ln|y| - \frac{\pi}{2}\, i$$

This complex logarithm satisfies, for non-zero z,

$$\log(1/z) \; = \; -\log z \quad \text{and} \quad \log(z_1 z_2) \; = \; \log z_1 + \log z_2 .$$

Examples of Complex Logarithms

1. $\log i \;=\; \{\,(2k+1/2)\pi\, i : k \in \mathbb{Z}\,\}$ so $\quad \text{Log (i)} \;=\; \frac{i\pi}{2}$
2. $\log{(1+i)} \;=\; \ln\sqrt{2} \;+\; (2k+1/4)\pi i$ for all k. Thus

$$\text{Log}(1+i) \;=\; \frac{1}{2}\ln 2 \;+\; \frac{\pi}{4} i.$$

3. If $r > 0$ then $\log{(-r)} \;=\; \ln r \;+\; (2k+1)\pi i$ so

$$Log(-r) \;=\; \ln r + \pi i$$

Let $\mathbb{R}_- := (-\infty, 0]$ be the closed negative real axis and $D := \mathbb{C} \backslash \mathbb{R}_-$. Then Log(z) is analytic on the domain D and

$$\frac{d}{dz}\text{Log}(z) \;=\; \frac{1}{z} \qquad \text{for } z \notin \mathbb{R}_-.$$

The real and imaginary parts of Log(z) are

$$u(x,y) := \ln|z| \text{ and } \quad v(x,y) := Arg(z).$$

u is harmonic on $\mathbb{C}\backslash\{0\}$, v is harmonic on D - but not continuous at any point in $\mathbb{R}_-$.

The real number x^α is defined whenever $x > 0, \alpha \in \mathbb{R}$ by

$$x^\alpha \ := \ e^{\alpha \ln x}$$

This is a real number and agrees with the other definitions when α is either an integer (in $\mathbb{Z}$) or a rational number (in $\mathbb{Q}$).

Complex Powers (Exponentiation)

We have seen that z^m is a well-defined complex number whenever $m \in \mathbb{N}$ and z^{-m} is well defined provided $z \neq 0$. Also there are m district possible values of $z^{1/m}$ if $z \neq 0$ - using the polar representation of z. Thus one can define $z^{m/n}$ when m, n are integers. In general z^α is defined by

$$z^\alpha \ := \ e^{\alpha \log z} \quad \text{when} \quad z \neq 0.$$

It has as many distinct values as $\log z$ has. That is a finite number of values when $\alpha \in \mathbb{Q}$, infinitely many values otherwise. The principal value of z^α is the value of the RHS of this formula with Log(z) in place of $\log z$.

The different values here are called different **branches** of the complex function of z^α .

The basic rules follow from the property of exponentials and logarithms. Namely

$$z^{\alpha_1} . z^{\alpha_2} \;=\; z^{(\alpha_1+\alpha_2)} \text{ and } \quad z^{-\alpha} \;=\; 1/z^{\alpha}.$$

Example 1. $\quad (-4)^{1/2} \;=\; \pm 2i$. This has two values and the principal value is $2i$.

Example 2. $\quad$ If γ is real then $i^{\gamma} \;=\; e^{\frac{\gamma \pi i}{2}}$. and $\log i = \frac{\pi}{2}\, i$

If $\gamma = \alpha + \beta i$ then $\gamma \log i \;=\; \frac{\pi}{2}\,(\alpha i - \beta)$ so

$$i^{\gamma} \;=\; e^{-\beta\pi/2} . e^{i\varphi} \qquad \text{with} \quad \varphi \;=\; \alpha\pi/2$$

The different values correspond to replacing β by other values here. Thus $i^{i} = e^{-\pi/2}$, or $e^{3\pi/2}$, or $e^{-5\pi/2}$ or $\ldots$.

Ex 3. $\quad (10i)^{i} \;=\; e^{-\pi/2 + i \ln 10} = e^{-\pi/2} . e^{i \ln 10}$

with $e^{i \ln 10} \;=\; \cos(\ln 10) + i \sin(\ln 10)$.

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3. Similar considerations apply to finding the inverse functions to $\sin z, \cos z, \tan z, \ldots, \sinh z, \cosh z, \ldots$.

Since all of these functions involve exponentials their inverse involve logarithms.