Introduction to Complex Analysis
Math 3364 Fall 2020

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The syllabus and course description are available on my web site and also at AccessUH. They describe how the course will be graded and when the midterm exam is. The final exam is as in the UH website.

This course will not follow any current text very closely. There are many texts available and they all cover the material in a similar way for the first few chapters. Nearly all the results to be described in this course were known by 1900 and many of the texts are quite similar. The most recent texts often include computational and graphical treatments of many topics - which shall not be covered here.

In the beginning there are the **integers**

$$\mathbb{N} \;:=\; \{1, 2, 3, 4, \ldots, n, \ldots\} \tag{1}$$

The set of **integers** and the set of **rational numbers** are

$$\mathbb{Z} \;:=\; \{0, \pm 1, \pm 2, \ldots, \pm n. \ldots\} \tag{2}$$

$$\mathbb{Q} \;:=\; \{\pm m/n : m \in \mathbb{Z}, \quad n \in \mathbb{N}\} \tag{3}$$

The set of **real numbers** is $\mathbb{R} \;:=\; (-\infty, \infty)$ The set of **extended real numbers** is $\overline{\mathbb{R}} \;:=\; [-\infty, \infty]$

$\pm\infty$ can be defined and are numbers (but not real numbers)

Complex numbers and functions are used throughout science and engineering - despite the fact that they are often called imaginary numbers.

A **complex number** is an expression of the form $x + iy$ where x, y are real numbers and i is a symbol to identify the number as being complex. Usually complex numbers are represented by w, z or a Greek lower case letter. When $z = x + iy$ then x = Re(z) the is called the **real part of z**, y = Im(z) is the **imaginary part of z**.

Two complex numbers are said to be equal if their real and imaginary parts are the same.

Algebra of Complex Numbers

The set of all complex numbers is denoted $\mathbb{C}$ and the complex numbers 0 and 1 are $0 = 0 + i\,0$ and $1 = 1 + i\,0$ Just as for real numbers, there are two basic operations; addition and multiplication defined by

Addition: $z_1 + z_2 \;:=\; (x_1 + x_2) + i\,(y_1 + y_2)$

Multiplication: $z_1.z_2 \;:=\; (x_1x_2 - y_1y_2) \;+\; i\,(x_2y_1 + x_1y_2)$

Note that addition is standard vector addition in $\mathbb{R}^2$ while multiplication is usual multiplication with the convention that $i^2 = -1$. It is different to the inner (or dot) product of vectors.

The inverse operations of subtraction and division that $-z = -x - iy$ and $w = z^{-1}$ are the complex numbers that satisfy

$$z \;+\; (-z) \;=\; 0 \quad \text{and} \quad z \cdot w \;=\; 1$$

$\mathbb{C}$ is a vector space and a commutative algebra with respect to the operations $+, .$ That is, both addition and multiplication obey the commutative and associative rules. Addition and multiplication satisfy the usual distributive rules of real numbers - so all your real algebra rules carry over to complex numbers and functions..

When $z = x + iy$ is a complex number then the **complex conjugate** of z is $\bar{z} := x - iy$.

The **modulus** of z is $|z| := \sqrt{x^2 + y^2}$ so $|z|^2 = z.\bar{z} = x^2 + y^2$. When $z \neq 0$ then the **multiplicative inverse** of z is the complex number w such that $w.z = z.w = 1$. Thus

$$z^{-1} := \frac{1}{z} := \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

Complex numbers are often regarded as points in the plane with Cartesian coordinates (x, y) so $\mathbb{C}$ is **isomorphic** to the plane $\mathbb{R}^2$. When $z = x + iy$ is a complex number then the **complex conjugate** of z is $\bar{z} := x - iy$. This is the reflection of a complex number z about the x-axis.

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There are many simple rules for these operations - section 1.2, page 5 of the text. The rules for conjugation and moduli also are quite simple. Complex numbers obey all the same algebraic propeties as real numbers - and this algebraic structure is called a **field** by mathematicians.

Exercise: For each of the following numbers z, evaluate $\overline{z}, |\, z\,|, 1/z$ and also evaluate various products.

$$1 + i, \quad 1 - i, \quad 3 + 4i, \quad \pm 1 + i\sqrt{3}, \quad \pm 5 + i\sqrt{12}$$

Do this witout calculators or computational assistance and then use some program to check your answers.

Most mathematical calculators and software packages have routines for complex arithmetic and I suggest that each of you learn to use such a calculator so you can check answers to homework problems.

Question: Do you have software on your computer that does complex arithmetic? Matlab, Mathematica, Maple, ...? Possibly Excel or other spreadsheet software ?

Strongly recommend that you find, and learn how to use, some software that does complex arithmetic.

Geometry of Complex Plane

When z is a **nonzero** complex number, then the **argument** of z is the angle θ where

$$\cos\theta \;=\; \frac{x}{|\,z\,|} \quad \text{and} \quad \sin\theta \;=\; \frac{y}{|\,z\,|}. \tag{4}$$

There is a unique value of theta in either $[0,\, 2\pi)$ or $(-\pi, \pi]$ such that (4) holds. (Radians are the units for angles here). From this equation you see that

$$z \;=\; |\,z\,|\; (\cos\theta + i\, \sin\theta)$$

and write $arg(z) := \theta$. This is called the **polar form** of the complex number z and represents z in terms of a length and an angle. Occasionally also "$cis\theta$" for $\cos\theta + i\,\sin\theta$.

The **triangle inequality** is the result that

$$|z_1 \; + \; z_2| \; \leq \; |z_1| \; + \; |z_2| \quad \text{for all} \;\; z_1, z_2 \in \mathbb{C}.$$

It is a nice exercise in geometry to prove this.

The polar form is most useful for multiplication. When z_1, z_2 are **nonzero** complex numbers with arguments θ_1, θ_2 then

$$z_1 \,.\, z_2 \;=\; |z_1| \,.\, |z_2| \; cis(\theta_1 + \theta_2)$$

This has a geometrical interpretation as $|z_1.z_2| = |z_1|.|z_2|$ and

$$arg(z_1 \,.\, z_2) \;=\; arg(z_1) + arg(z_2)$$

$$arg\left(\frac{z_1}{z_2}\right) \;=\; arg(z_1) \;-\; arg(z_2)$$

There are lots of amazing formulae involving complex numbers and functions - and they may be used in many different ways. Many geometrical questions also can be viewed using complex geometry so we will look at some of them.

The Exponential function

The ordinary exponential function is the function $f(x) := e^x$. It is the unique solution of the equation

$$\frac{df}{dx} = f(x) \qquad f(0) = 1$$

and has the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \frac{x^m}{m!} + \ldots .$$

This series converges for every real $x \in \mathbb{R}$ and the function has the property $e^{x_1 + x_2} = e^{x_1} . e^{x_2}$

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Note that if a is a positive real number the general exponential function is $g(x) = a^x$ and has $g'(x) = \ln a . a^x$.

The Complex Exponential function

All the preceding formulae still hold if x is replaced by the complex number $z \in \mathbb{C}$. In particular define

$$e^z \;:=\; e^x \,.\, e^{yi} \quad \text{with}$$

$$e^{yi} \;=\; 1 \;+\; yi \;-\; \frac{y^2}{2!} \;+\; \ldots \;+\; \frac{(iy)^m}{m!} \;+\; \ldots .$$

From the Taylor series for sin and cosine functions, this becomes

$$e^z \;:=\; e^{x+\,iy} \;=\; e^x\,(\cos y \;+\; i \sin y).$$

Note that $e^0 = 1$ so $e^{0i} = 1$ when $y = 0$ so this complex definition agrees with the definition that was given in Calc I.

Thus $e^{i\theta} \; = \; \cos\theta \; + \; i \; \sin\theta$ and the **polar representation** of complex numbers

$$z \; = \; |z| \; e^{i\theta} \; = \; r \; e^{i\theta}$$

Also one finds the formulae

$$\cos\theta \; = \; \frac{e^{i\theta} \; + \; e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta \; = \; \frac{e^{i\theta} \; - \; e^{-i\theta}}{2i}$$

When $\quad z_1 = r_1 \, e^{i\theta_1}, \; z_2 = r_2 \, e^{i\theta_2} \quad$ then

$$z_1 . z_2 \; = \; r_1 \, r_2 \, e^{i(\theta_1 + \theta_2)} \quad \text{and} \quad \frac{z_1}{z_2} \; = \; \frac{r_1}{r_2} \, e^{i(\theta_1 - \theta_2)}$$

provided $r_2 \neq 0$.

The preceding results lead to **DeMoivre's theorem**

$$\cos m\theta + i \sin m\theta \ = \ e^{im\theta} \ = \ (\cos\theta + i\sin\theta)^m$$

These enable the derivation of formulae for cos of $m\theta$ in terms of cos of θ

$$\cos m\theta = T_m(\cos\theta)$$

T_m is the m-th Chebychev polynomial and is a polynomial of degree m. Unfortunately $\sin m\theta$ is not a polynomial in $\sin\theta$. (Check when m =2,3.)

Example. The quartic expansion is
$(a+b)^4 \ = \ a^4 + 4a^3b + 6a^2\,b^2 + 4ab^3 + b^4$. Then
$\cos 4\theta \ = \ (\cos\theta)^4 - 6(\cos\theta)^2(\sin\theta)^2 + (\sin\theta)^4$
by equating real and imaginary parts. Thus
$\cos 4\theta \ = \ 8(\cos\theta)^4 - 8(\cos\theta)^2 + 1.$

Euler's formula has $x = 0, y = \theta$ in the definition of e^z, so

$$e^{i\theta} \;=\; \cos\theta \;+\; i\;\sin\theta$$

See Wikipedia for more about the history. When θ is $\pi/2,\;\pi, 3\pi/2$ respectively we have

$$e^{i\pi/2} \;=\; i, \qquad e^{i\pi} \;=\; -1, \qquad e^{3i\pi/2} \;=\; -i$$

These are Euler's identities and

$$e^{i(\theta+2k\pi)} \;=\; e^{i\theta} \qquad \text{for all} \;\; k \in \mathbb{Z}$$

That is $e^{i\theta}$ is periodic of period 2π. A function is periodic of period T provided $f(t + T) = f(t)$ for all $t \in \mathbb{R}$ or $f(t + kT) \;=\; f(t)$ for all t and $k \in \mathbb{Z}$.

Roots of Unity

Problem: Find the solutions of $z^m = 1$ when $m \in \mathbb{N}$.
Solutions of this equation are called the **m-th roots of unity.**

When m =2, the solutions of $z^2 = 1$ are $z = \pm 1$ since

$$z^2 - 1 = (z-1).(z+1).$$

When $m = 3$ one has $z^3 - 1 = (z-1).(z^2+z+1)$.
For any $m \geq 2$, one has the geometric series

$$z^m - 1 = (z-1).(1 + z + z^2 + \ldots + z^{m-1})$$

so $z = 1$ is always a solution and $z = -1$ is also a solution when m is even. These are the only possible real solutions.

From the Euler identities $e^{2ki\pi} = 1$ for any integer k. Thus $\omega_m := \exp 2i\pi/m$ satisfies $\omega_m^m = 1$.

Similarly ω_m^k satisfies the equation for any integer k. However only the points where $k = 0, 1, 2, \ldots, m-1$ are different - so there are m distinct complex m-th roots of unity. Thus when $m \geq 3$ there always are complex roots as well as the real roots.

Example. the solutions of $z^4 - 1 = 0$ are $z = \pm 1, \pm i$ as $(z^4 - 1) = (z^2 - 1).(z^2 + 1)$.

In general there will be m distinct solutions of $z^m = a$ when a is a non-zero complex number.

Suppose $a = |a|e^{i\theta}$. Then the solutions are

$$z_k := |a|^{1/m} e^{i(\theta+2k\pi)/m} \qquad \text{for } k = 0, 1, \ldots, m-1$$

The solutions of the quadratic equation

$$az^2 + bz + c = 0 \quad \text{with } a \neq 0 \quad \text{are}$$

$$z_{\pm} := \frac{1}{2a} \left[-b \pm [b^2 - 4ac]^{1/2} \right]$$

where the last term is taken to be a complex square root.

If a polynomial $p(z) = z^m + a_1 z^{m-1} + \ldots + a_{m-1} z + a_m$ can be factored as the product of other polynomials $p(z) = p_1(z).p_2(z)$ then the solutions of $p(z) = 0$ are either solutions of $p_1(z) = 0$ or of $p_2(z) = 0$.

Sets of Complex Numbers

The **open disk center** z_0 and radius ρ is the set

$$B_\rho(z_0) \; := \; \{z \in \mathbb{C} : |z - z_0| < \rho\}$$

This set is often called the ρ neighborhood of z_0. A set S is said to be **bounded** if there is an R such that $S \subset B_R(0)$.

When S is a set of complex numbers then z_0 is an **interior point** of S provided there is a $\rho > 0$ such that $B_\rho(z_0) \subset S$.

S is said to be an open set if for each point of S is an interior point.

The **closed disk center** z_0 and radius ρ is the set

$$\overline{B}_\rho(z_0) \; := \; \{z \in \mathbb{C} : |z - z_0| \leq \rho\}$$

When z_0, z_1 are complex numbers then the **interval** $[z_0, z_1]$ is the straight line joining these endpoints. That is,

$$[z_0, z_1] \ := \ \{(1-t)z_0 + tz_1 : 0 \leq t \leq 1\}$$

A **polygonal path** with nodes $z_0, z_1, z_2, \ldots, z_m$ is the union of the straight lines joining these points (in order).

$$\text{Path} \ := \ [z_0, z_1] \cup [z_1, z_2] \cup \ldots [z_{m-1}, z_m]$$

A (non-empty) subset S of $\mathbb{C}$ is said to be **connected** if any two points w, z in S may be joined by a polygonal path that is a subset of S.

A (non-empty) open connected subset S of $\mathbb{C}$ is called a **domain**.

The complement of a set S is the set $\tilde{S} := \mathbb{C} \backslash S$ of all complex numbers that are not in S.

A point $z_0 \in \mathbb{C}$ lies in the boundary of a (nonempty) set S provided every open disk centered at z_0 contains points in S and also points in $\tilde{S}$.

The **boundary of S** is the set of all boundary points of S and is denoted ∂S. The set S is said to be closed if it contains all of its boundary points.

A set that is bounded and closed is said to be **compact**. A set that is not bounded is said to be **unbounded**.