## Complex Analysis

For the fall 2020 midterm exam, the material covered many of the basic results about complex numbers, and the properties of differentiable complex functions. Then a number of weeks were spent on the theory of contour integrals, the Cauchy-Goursat theorem, the Cauchy integral formula and its generalizations and related results.

The theory of Taylor and Laurent approximations of functions was then described - but was not on the exam. Now we shall describe various applications and further results about analytic functions on domains in $\mathbb{C}$. First we shall describe more results about complex polynomials and the solutions of polynomial equations.

Finding solutions of Polynomial Equations

One of the oldest problems in mathematics was to find solutions of equations of the form

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n \;=\; 0$$

Here n is called the degree of the polynomial.

Here the coefficients $a_1, \ldots a_n$ may be required to be rational, real or general complex numbers. A real number $\hat{z}$ is said to be an algebraic number if it is a solution of an equation of this form with each $a_j$ rational.

The numbers $\pi, e$ and many others are known not to be algebraic numbers, but the numbers $\sqrt{2}, \sqrt{3}$ are algebraic but not rational numbers. There are explicit formulae for the solutions fo equations of degree 2, 3 and 4 but Galois proved there was no formula for some polynomials of degree 5 of more.

The importance of complex numbers is that if the coefficients of a polynomial are complex numbers, a polynomial equation has n complex solutions - if you count appropriately. This is the **Fundamental theorem of Algebra**.

We shall prove this and study various related results. First we shall write

$$p(z) \ := \ z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n$$

and observe that this is an analytic function of z on $\mathbb{C}$. It has derivatives of all orders and the k-th derivative $p^{(k)}(z)$ will be a polynomial of degree $n-k$ in z for $1 leqsk \ \leq \ n$ and its is identically zero for $k > n$.

One place where one obtains polynomial equations is in finding eigenvalues of an $n \times n$ real or complex matrix A. The eigenvalues will be the solutions of the equation

$$det(\lambda I_n - A) \;=\; 0$$

This will be a polynomial equation with $a_1 = -trA,\; a_n = -detA$.

Similar polynomial equations often arise in engineering and in finance in connection with iterative processes and growth.

First we shall show how to reduce the order of a polynomial when one knows a zero. Suppose that $\zeta$ is a solution of $p(z) = 0$ then we want to find the polynomial $q(z)$ such that

$$q(z) := z^{n-1} + b_1 z^{n-2} + b_2 z^{n-3} + \ldots + b_{n-2} z + b_{n-1}, \quad \text{and}$$

$$p(z) \;\; := \;\; (z \; - \; \zeta) \; q(z).$$

The formulae for the coefficients $b_j$ in terms of the $a_j$ is that

$$b_1 = a_1 + \zeta \quad \text{and} \qquad b_j = a_j + \zeta \, b_{j-1} \qquad \text{for} \quad 2 \leq j \leq n-1.$$

One also must have $a_n = -\zeta \, b_{n-1}$.

This is proved by multiplying out the expression $(z - \zeta) \, q(z)$ and equating coefficients.

These equations are usually solved recursively. That is given $\zeta$, evaluate $b_1, b_2, b_3, \ldots, b_{n-1}$ in order from these equations. Alternatively use the last formula for $b_{n-1}$ and then determine $b_{n-2}, \ldots, b_1$ in reverse order.

This is a linear system of $n-1$ equations for the $b_j$ in terms of the $a_j$.

I suggest that you write this as a matrix equation $Lb = a$ where $a, b$ are column vectors involving the $a_j$, $b_j$ respectively. Find the inverse $L^{-1}$ of this matrix when $n = 2, 3$ and 4.

A function $f$ on $\mathbb{C}$ is said to be **entire** provided it is analytic for all z.

Such a function is analytic on every disk in the complex plane so the bounds derived from Cauchy's integral theorem lead to the following famous theorem

**Theorem** (Liouville) If an entire function is bounded on $\mathbb{C}$, then it is constant on $\mathbb{C}$.

When $p(z)$ is a polynomial of degree n of the form

$$p(z) \ := \ a_0 z^n \ + \ a_1 z^{n-1} \ + \ \ldots + a_{n-1} z \ + \ a_n$$

with $a_0 \neq 0$ and each $a_j \in \mathbb{C}$. Then

$$\lim_{|z| \to \infty} \frac{p(z)}{z^n} \ = \ a_0.$$

Supppose that we now that every polynomial of degree n has at least one complex zero. If we know one complex zero $\zeta_1$ then the factorization algorithm yields

$$p(z) = (z - \zeta_1)(a_0 z^{n-1} + b_1 z^{n-2} + \ldots + b_{n-2} z + b_{n-1}$$

where you can evaluate the new coefficients $b_j$.

Thus $p(z) = (z - \zeta_1) q_1(z)$ with $q(z)$ a polynomial of degree $(n-1)$ and leading coefficient $a_0 \neq 0$. Apply the theorem again to find a second zero $\zeta_2$ and $p(z)$ is the product of two factors $(z - \zeta_1)(z - \zeta_2)\, q_2(z)$ with $q_2$ a polynomial of degree $(n-2)$. Continue until you have a product of n simple factors $(z - \zeta_j)$ - so the $\zeta_j$ will be the zeroes of the polynomial $p(z)$. This is called factoring a polynomial.

A zero $\zeta_j$ of $p(z)$ has multiplicity m if $p(z) = (z - \zeta_j)^m q(z)$ with $q(\zeta_j) \neq 0$. Then there will be at most n distinct complex zeroes of $p(z)$ and the number of zeros counting multiplicity is n.

So to prove the fundamental theorem of algebra it suffices to prove that every polynomial has one complex zero as we have just shown that if that holds then it has n complex zeroes by repeated factorization.

Example  Factor $p(z) = z^4 - 2z^3 + 2z^2 - 2z + 1$ and determine the number of distinct zeros and their multiplicity.

## The Fundamental Theorem of Algebra.

**Theorem** A complex polynomial of degree n has a complex zero.

This holds by using Liouville's theorem on the function

$$F(z) := \frac{1}{p(z)}$$

If $p(z)$ did not have a zero, the $F(z)$ is an entire function on $\mathbb{C}$. So it must be a constant and p(z) must be constant. Thus degree p $= 0$ which contradicts our assumption that degree p $= n \geq 1$

## The Calculus of Residues

Every text on mathematical physics has a section on the calculus of residues because it is a way of finding formulae for loop integrals of analytic functions that have a physical interpolation, Often there is no other way of calculating quantities such as circulation or the number of solutions of an equation except by using the residue theorem. Even numerical methods can be tough. (except maybe numerically).

First assume that we are given a function $f(z)$ which is analytic on a domain D and has a finite number of isolated singularities at points $\mathcal{S} := \{\zeta_j : 1 \le j \le J\}$ that are in $\overline{D}$.

Suppose $\zeta$ is an isolated singularity of $f(z)$ and $f$ is analytic on and inside a simple loop $\Gamma$ except at the singularity $\zeta$. Then $f(z)$ has a Laurent series representation in a neighborhood of $\zeta$. That is

$$f(z) = \lim_{M \to \infty} \sum_{k=-M}^{M} a_k \, (z - \zeta)^k$$

and this finite approximations will converge to $f(z)$ on a deleted disk $A = \{z : 0 < |z - \zeta| < r\}$ for $r < R$.

Let $\Gamma$ be a simple positively oriented loop in A and $C$ be a positively oriented circle around $\zeta$ and inside $\Gamma$. Then the homotopy theorem and a formula for integrating these functions around a circle yield

$$\int_\Gamma f(z)\, dz \;=\; \int_C f(z)\, dz \;=\; 2\pi\, i\, a_{-1}$$

since all the other loop integrals

$$\int_C (z-\zeta)^k\, dz \;=\; 0 \qquad \text{when } k \neq -1.$$

Thus the integral around $\Gamma$ is given by

$$\int_\Gamma \; f(z)\, dz \;=\; 2\pi\, i\, a_{-1}$$

for any function that is analytic in $D\backslash\{\zeta\}$.

The quantity $a_{-1}$ is called the **residue of f at** $\zeta$. It is a particular coefficient in the Laurent approximation of $f(z)$ near $\zeta$. and is denoted **Res(f;** $\zeta$**)** .

When $f(.)$ has a simple pole at $\zeta$, then the residue is

$$Res(f;\zeta) \;=\; \lim_{z\to\zeta}\,(z-\zeta)\,f(z)$$

In particular if $f(z) = p(z)/q(z)$ and $\zeta$ is a simple zero of $q(z)$, then

$$Res(f;\zeta) \;=\; \frac{p(\zeta)}{q'(\zeta)}$$

where $q'(z)$ is the derivative of $q(z)$.

Assume for simplicity that $q(z)$ is a polynomial, then from the factorization theorem one has

$$q(z) \;=\; (z-\zeta)\; q_1(z) \quad \text{with } q_1(\zeta) \neq 0 \quad \text{so}$$

$$\lim_{z\to\zeta}(z-\zeta)\, f(z) \;=\; \lim_{z\to\zeta} \frac{p(z)}{q_1(z)} \;=\; \frac{p(\zeta)}{q_1(\zeta)}$$

as $p, q_1$ are continuous at $\zeta$ and $q_1(\zeta) \neq 0$. This is the desired formula for the residue as $q'(\zeta) \;=\; q_1(\zeta)$. If $q(z)$ is analytic near $\zeta$ but not a polynomial you use some properties of the Taylor approximations to prove this still holds.

Similarly, when $f(.)$ has a pole of order m at $z_0$, then the residue is

$$Res(f : z_0) \;=\; \lim_{z\to z_0} \frac{1}{(z-z_0)^m}\, \frac{d^{m-1}}{dz^{m-1}}\, (z-z_0)^m\, f(z)$$

he calculus of residues is based on the following theorem which gives the value of a contour integral when $f(z)$ only has isolated singularities inside $\Gamma$.

**Theorem** (Cauchy) Suppose $\Gamma$ is a simple, positively oriented loop (spol) and $f(z)$ is analytic inside and continuous on $\Gamma$, except at a finite number of isolated singularities at $\zeta_1, \ldots, \zeta_J$ inside $\Gamma$. Then

$$\int_\Gamma f(z)\, dz \;=\; 2\pi\, i \sum_{j=1}^{J} Res(f; \zeta_j)$$

Note that if there is only a single singularity inside $\Gamma$ this is just the Cauchy integral formula. The proof of this theorem is based on showing that this integral equals the sum of the integrals around the individual singularities. That is the value of this integral is $2\pi\, i$ times the sum of the residues of $f$ at its isolated singularities.

# Evaluation of some Trigonometric Integrals

Consider the problem of evaluating

$$\int_0^{2\pi} g(\cos\theta, \sin\theta)\ d\theta$$

where $g$ is an analytic function on the square center 0 and sides of length 2. The functions $\cos\theta, \sin\theta$ are the values of the functions

$$f_1(z) := \frac{1}{2}(z + z^{-1}) \text{ and } \quad f_2(z) := \frac{1}{2i}(z - z^{-1})$$

on the unit circle C. ... (Write down a formula for points in C and substitute.) Define

$$G(z) \ := \ \frac{1}{i\,z}\, g\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)$$

Example: Evaluate

$$I(a) \; := \; \int_0^{2\pi} \frac{d\,\theta}{2 - a\cos\theta} \quad \text{with } |a| < 2.$$

Here $g(\cos\theta) \; = \; [2 - (a/2)\,(z + z^{-1})]^{-1}$ . Let $a = 1$, then

$$I(1) \; = \; \frac{2}{i} \int_C \frac{dz}{4z - z^2 - 1}$$

where C is the unit circle. One has that

$$(z^2 \; - \; 4z \; + \; 1) \; = \; (z \; - \; z_+)\,(z - z_-)$$

where $z_\pm \; := \; 2 \pm \sqrt{3}$. Thus $z_\pm$ are simple poles of this integrand and only $z_-$ is inside the unit circle.

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where $z_\pm \;:=\; 2 \pm \sqrt{3}$. Thus $z_\pm$ are simple poles of this integrand and only $z_-$ is inside the unit circle. The residue of $G$ at $z_-$ is $(-1)/(2\sqrt{3})$ so one finds that

$$I(1) \;=\; \frac{-4\pi i}{i}\,\frac{-1}{2\sqrt{3}} \;=\; \frac{2\pi}{\sqrt{3}}.$$

For more general a, the quadratic equation changes some (please check) - but one can still evaluate this integral explicitly.

A similar, famous, formula is that

$$\int_0^{2\pi} \frac{d\,\theta}{a^2\, sin^2\theta \;+\; b^2\, cos^2\theta} \;=\; \frac{2\pi}{ab} \quad \text{when } a, b > 0.$$

Note that if either a or b is zero then this right hand side is $\infty$ and the identity still holds. With this you can evaluate integrals such as

$$\int_0^{2\pi} \frac{d\,\theta}{1 \;\pm\; c^2\; sin^2\theta} \quad \text{with } c^2 < 1.$$

Please try. These all are examples of problems where we evaluate integrals without knowing an antiderivative for most values of the parameters!

The general case is based on determining a function G such that

$$\int_0^{2\pi} g(\cos\theta, \sin\theta)\; d\,\theta = \int_C \; G(z)dz.$$

with C the unit circle in the complex plane.

Then find the singularities of $G$ inside C. If these are isolated singularities, evaluate the residues of G at these poles. Finally use the residue theorem to evaluate the integral.

Most complex analysis textbooks provide a variety of examples either as worked problems or exercises.

Suppose $f(z)$ is a function that is analytic on a domain D except possibly for isolated poles in D. Such a function is called **meromorphic on D**

When $f(z)$ has a zero of order m at a point $z_0$, then $f(z) = (z - z_0)^m g(z)$ with g analytic near $z_0$ and $g(z_0) \neq 0$. This follows from Taylor's theorem and $z_0$ will be said to be a **zero of multiplicity m.**

When $f(z)$ has a pole of order m at a point $z_0$, then

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{with} \quad g(z_0) \neq 0$$

and g analytic near $z_0$. Such a $z_0$ is said to be a **pole of multiplicity m.**

Consider the function $F(z) := \frac{f'(z)}{f(z)}$. This called the **logarithmic derivative** of $f$ and will have poles at the zeroes of $f(z)$ and zeroes at the poles of $f(z)$ and the zeroes of $f'(z)$.

When $z_0$ is a zero of order m of f, and $f(z) = (z - z_0)^m g(z)$, then $f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)$ so

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

with $g(z) \neq 0$ close to $z_0$. Thus $F(z)$ has a simple pole at $z_0$ and its residue at the pole is m, So the contour integral of $F(z)$ around a p.o. simple loop containing $z_0$ and no other zero or singularity will be $2\pi i m$, When $\zeta_0$ is a pole of order m of $f(z)$ then $f(z) = \frac{g(z)}{(z-\zeta_0)^m}$ implies that

$$f'(z) \;=\; -m(z-\zeta_0)^{-(m+1)}\,g(z) \;+\; (z-\zeta_0)^{-m}\,g'(z) \quad \text{so}$$

$$F(z) \;=\; \frac{-m}{z-\zeta_0} \;+\; \frac{g'(z)}{g(z)}$$

and again $F$ has a simple pole at $\zeta_0$ with residue $-m$.

Suppose $f(z)$ is meromorphic inside a simple p.o. loop $\Gamma$ and continuous and non-zero on $\Gamma$ with zeroes inside $\Gamma$ at $S_z := \{z_j; 1 \leq j \leq J\}$, and poles at

$S_p \;:=\; \{\zeta_k; 1 \leq k \leq K\}$, then $F(z)$ will be analytic on this region except for simple poles at the points in $S_z \cup S_p$.

Suppose that $m_j$ is the multiplicity of $z_j$ as a zero, and $m_k$ is the multiplicity of $\zeta_k$ as a pole. Define

$$N_z(f) \;:=\; \sum_{j=1}^{J} m_j \text{ and } \quad N_p(f) \;:=\; \sum_{k=1}^{K} m_k.$$

The following theorem that is known as the **principle of the argument**.

**Theorem:** If $f(z)$ is meromorphic inside a simple p.o. loop $\Gamma$ and analytic and nonzero on $\Gamma$ then

$$N_z(f) \;-\; N_p(f) \;=\; \frac{1}{2\pi\, i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz$$

The proof of this result is based on showing that this loop integral equals the sum of loop integrals around each pole and each zero of F(z). These are $m_j, m_k$ respectively so the result follows by carefully showing how to replace the integral around Γ by complicated curves that just include 1 singularity of F at a time. A special case is the following.

**Corollary:** If $f(z)$ is analytic inside and on a simple loop C and nonzero on C, then

$$N_z(f) \;=\; \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz$$

So if we want to know how many zeroes of a polynomial lie in a specific region of the complex plane one need only evaluate a contour integral around the boundary of the region. This usually is a simple issue to do after you choose a loop that doesn't pass through any zero.

Numerically one does not have to compute these integrals very accurately since their values must be integers. If one uses approximate numerical integration and gets an answer of $3.2 \pm 0.4$, then the actual value has to be 3. It used to be that computing an integral was easier than solving equations but the following result has always been used extensively.

In practice, however, one often simplifies the problem by replacing the function $f(z)$ by a simpler function for which the answer is even easier. This is done using the following result.

**Theorem** (Rouché ) Suppose that $f, h$ are analytic functions inside and on a simple p.o. loop C. If $|h(z)| < |f(z)|$ on C then $f, f + h$ have the same number of zeros inside C.

Example. Show that there are 4 solutions of the equation
$6z^4 + z^3 - 2z^2 + z = 1$ inside the unit disk $B_1$.

Take this function to be the $f(z) + h(z)$, and choose a simple function $f(z)$ such that the theorem can be used. Then show that the difference $h(z)$ obeys the conditions of the theorem.

When $\Gamma$ is a closed loop in the complex plane and $z_0 \notin \Gamma$ then the **winding number (or index) of $\Gamma$ about $z_0$** is the value of the integral

$$Ind_\Gamma(z_0) \; := \; \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - z_0}$$

This integral is always an integer ( possibly negative).

For a simple loop, this number is either $\pm 1$, negative when the curve goes clockwise. When $\Gamma$ is not be simple, but may be parametrized by differentiable functions except at a finite number of corners, then this number will be finite if the loop has finite length.

It counts the number of times a closed curve "winds around" a point $z_0$.

If a closed contour is not simple then all the integral formulae the winding number must be included in contour integral formulae.