Evaluating Infinite Integrals and Transforms

Very often in this course and elsewhere we need to define and evaluate integrals over either the whole real line, the interval $(0, \infty)$, or over "infinite contours".
Perhaps the most famous such integrals are formulae such as

$$\int_{-\infty}^{\infty} e^{-a x^2} \, dx \; = \; \sqrt{\pi / a}$$

for the Gaussian function.

In many applications of mathematics we use Fourier and Laplace transforms. The **Laplace transform** of a function f defined on $[0, \infty)$ is

$$\mathcal{L}f(s) \;:=\; \int_0^\infty \; e^{-st}\, f(t)\, dt \quad \text{with } s > 0.$$

The **Fourier transform** $\mathcal{F}f$ of a function f defined on $\mathbb{R}$ is

$$\mathcal{F}f(k) \;:=\; \int_{-\infty}^\infty \; e^{-ikx}\, f(x)\, dx \;=\; g(k) \qquad k \in \mathbb{R}.$$

When the Fourier transform $g(k) = \mathcal{F}f(k)$ is known then the inverse Fourier transform is

$$f(x) \;:=\; (\mathcal{F}^{-1}g)(x) \;:=\; \frac{1}{2\pi} \int_{-\infty}^\infty \; e^{ikx}\, g(k)\, dk$$

This is one of the great formulae in mathematics that most graduate students in analysis will use sometime.

The integration theory that you learnt in Calculus 1 and 2 only works for integrals over finite closed intervals $[a, b]$. Many results must be changed if the function to be integrated is only continuous on an open interval (a, b). Consider the problem of evaluating

$$\int_0^1 f(x)\,dx \quad \text{with } f(x) = x^{-\alpha}$$

This function f is always continuous on (0,1) but when $\alpha \geq 1$, the integral is ∞ - or not defined. That is you cannot always integrate a continuous function on an open interval.

Intervals over infinite sets must be approximated by integrals over finite intervals such as [0,R] or [-R,R]. Then you must prove that the limits as $R \to \infty$ exist. If so, these limit will be the integrals over the infinite integral.
Thus for the Gaussian integral a mathematician has to show that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-a\,x^2}\, dx \;=\; \sqrt{\pi/a},$$

Note that in this case one has that $e^{-a\,x^2} \to 0$ very rapidly for $|x|$ large, so this is an easy limit. Many others are not so easy. The currently used definitions of the Fourier transform (FT) were only developed in the 1950's - even though FTs had been used by everyone for over 120 years.

Probability Density Functions

The ubiquitous Gaussian function is $f(x,a) := e^{-a x^2}$.
This function is everywhere positive and there is a constant c_a such that $\rho(x) := c_a\, f(x,a)$ is a probability density function. That is $\rho(x) \geqslant 0$ for all x and the area under the curve $y = \rho(x)$ satisfies

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1$$

To find the appropriate constant c_a one needs to evaluate

$$M(a) := \int_{-\infty}^{\infty} f(x,a)\, dx.$$

then choose $c_a := 1/M(a)$. For the Gaussian, Wikipedia has a proof that $c_a = \sqrt{a/\pi}$ using Calculus 3.

There are many important probability density functions that arise from functions of the form

$$f(x,a) \;:=\; \frac{p(x,a)}{q(x,a)}$$

where $p(.,a), q(.,a)$ are positive polynomials on $(-\infty,\infty)$. An example is $f(x,a) \;:=\; 1/(x^2+a^2)$. This has

$$M(a) \;:=\; \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a} \quad \text{so} \;\; \rho(x) \;:=\; \frac{a}{\pi\,(x^2+a^2)}.$$

is a probability density function for any positive a.

You may know an the antiderivative of $1/(x^2 + a^2)$ but there are only a few such formulae known.
To illustrate this consider the problem of evaluating the integrals of $f(x, a) := 1/(x^4 + a^4)$.
Instead of evaluating this real integral, consider the problem of evaluating the contour integral of $f(z) := 1/(z^4 + a^4)$ along a contour Γ_R that goes from -R to R along the real axis, then is given by $\zeta(t) = R\, e^{it}$ with $0 \leq t \leq \pi$. This is a semicircle C_R of radius R center the origin from R to -R.
Γ_R is a simple loop that satisfies the conditions needed for Cauchy's formulae and the caculus of residues.

Then the contour integral is

$$\int_{\Gamma_R} f(z)\ dz \ = \ \int_{-R}^{R} f(x)\ dx \ + \ \int_0^{\pi} \ f(Re^{it})(iRe^{it})\, dt$$

From the calculus of residues one has that, since $f(z)$ only has poles, this integral

$$= \ 2\pi i \,[\text{sum of residues of f(z) inside } \Gamma_R]$$

To evaluate this, first need to find the poles of $f(z)$. They are the solutions of $z^4 + a^4 = 0$, so one finds that they are

$$z_1 \ = \ a\, e^{i\pi/4}, z_2 \ = \ a\, e^{3i\pi/4}, z_3 \ = \ a\, e^{5i\pi/4}, z_4 \ = \ a\, e^{7i\pi/4}$$

Only the first two are in the upper half plane and the residues of the function at these simple poles are

$$a_1 \;:=\; \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \;=\; \frac{-(1+i)}{4\sqrt{2}\;a^3}$$

$$a_2 \;:=\; \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} \;=\; \frac{1-i}{4\sqrt{2}\;a^3}$$

respectively. Thus, when $R > a$, so the poles are inside the contour,

$$a_1 \;+\; a_2 \;=\; \frac{-i}{2\sqrt{2}\;a^3} \qquad \text{and}$$

$$\int_{\Gamma_R} f(z)\;dz \;=\; \frac{\pi}{\sqrt{2}\;a^3}$$

A probabilist will now conclude that

$$M(a) \ := \ \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} \ = \ \frac{\pi}{\sqrt{2}\ a^3}, \text{ and}$$

$$\rho(x) \ = \ \frac{\sqrt{2}\ a^3}{\pi\,(x^4 + a^4)}$$

is a probability density function.
Why can they do this? First note that the contour integral does not change provided $R > a$. So for all $R > a$ we see that

$$\int_{-R}^{R} f(x)\ dx \ + \ \int_0^{\pi} \ f(Re^{it})(iRe^{it})\,dt \ = \ \frac{\pi}{\sqrt{2}\ a^3}$$

The following elementary result is a good exercise.
Lemma Suppose that $f(z) = p(z)/q(z)$ is the ratio of two polynomials and degree of q - degree p ≥ 2. If Γ_R is a subinterval of the circle of radius R center the origin with $\theta \in [\theta_1, \theta_2]$, then $|f(z)| \to 0$ as $|z| \to \infty$ and

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{p(z)}{q(z)}\, dz \;=\; 0$$

Use this in the preceding problem. One has degree q - degree p $=$ 4, so

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \;=\; \frac{\pi}{\sqrt{2}\ a^3} \;-\; \lim_{R\to\infty} \int_0^{\pi} f(Re^{it})(iRe^{it})\, dt.$$

Since this last limit is zero, the formulae for the integrals given above hold.

Laplace Transforms and their Inverses

In your first course in ordinary differential equations, you may have used Laplace transforms to solve problems for linear equations with constant coefficients. Those methods were based on an "operational calculus" for solving differential equations in circuit theory due primarily to Oliver Heaviside in the late 19th century. The **Laplace transform** of a function $f : [0, \infty) \to \mathbb{R}$ is defined to be the integral

$$\mathcal{L}f(s) \ := \ \int_0^\infty \ e^{-st} \ f(t) \ dt$$

The function $f(t)$ here could be complex valued. When $s = i\xi$, this is a Fourier transform as

$$\mathcal{L}f(i\xi) \ := \ \int_0^\infty \ e^{-i\xi t} \ f(t) \ dt$$

The definition of the Laplace transform is really that

$$\mathcal{L}f(s) \ := \ \lim_{R \to \infty} \int_0^R \ e^{-st} f(t)\, dt$$

The finite integral $F_R(s) \ := \ \int_0^R \ e^{-st} f(t)\, dt$ is a simple integral when f is continuous on $[0, R]$. $F_R(s)$ is a function of s that has continuous m-th derivatives for every integer m with

$$F_R^{(m)}(s) \ = \ \int_0^R \ t^m e^{-st} f(t)\, dt$$

Simple examples include the Laplace transform (LT) of $e^{-at}, t^m, \sin \omega t$ and $\cos \omega t$ respectively are

$$\frac{1}{s+a}, \quad \frac{m!}{s^{m+1}}, \quad \frac{\omega}{(s^2 \ + \ \omega^2)}, \quad \frac{s}{(s^2 \ + \ \omega^2).}$$

The delta function $\delta_a(t)$ has Laplace transform e^{-as}.

These Laplace transforms are analytic functions with simple poles. Sometimes the calculus of residues is used to evaluate Laplace transforms of rational functions $f(t) = p(t)/q(t)$. Consider the contour integral of $e^{-sz} f(z)$ along a contour Γ_R that goes from the origin to R along the x-axis, takes a quarter circle of radius R to iR and then comes down along the y-axis to the origin. Then evaluate the integral

$$\int_{\Gamma_R} e^{-sz} f(z)\, dz$$

by the calculus of residues and find out what happens as $R \to \infty$. More often, however, one is given a Laplace transform $F(s)$ and wants what the function $f(t)$ was.

From the differential equations you generally have a formula for the Laplace transform of the solution so one wants to find the function $f(t)$ with that transform. That is you are are given some function F(s) and want to find the solution $f(t)$ of the first type integral equation

$$\int_0^\infty e^{-st}\, f(t)\, dt \;=\; F(s)$$

This is a badly ill-posed (or unstable) numerical problem. First note that if $|f(t)| \leqslant Me^{at}$, then $|F(s)| \leqslant M/(s-a)$ when s is real and larger than a. You can only define Laplace transforms that satisfy some such growth condition.
When s is complex, $Re(s) > s_0$, for these functions f, a similar inequality show that $F(s)$ is a bounded function on the half-plane where $Re(s) > s_0$.

So if $F(s)$ has any singularities they must lie in the half plane where $Re(s) \leqslant s_0$. When a function f has derivatives, their Laplace transforms are

$$\mathcal{L}(f')(s) \ = \ sF(s) - f(0), \quad \mathcal{L}(f")(s) \ = \ s^2 F(s) - s\, f(0) - f'(0),$$

$$\mathcal{L}(f''')(s) \ = \ s^3 F(s) - s^2 f(0) - s\, f'(0) - f''(0).$$

Suppose f satisfies the linear 3rd order ordinary differential equation with constant coefficients,

$$f'''(t) \ + \ a_1 f''(t) \ + \ a_2 f'(t) \ + \ a_3 f(t) \ = \ g(t) \quad \text{for} \quad t > 0$$

then its Laplace transform satisfies

$$Q(s)F(s) \ := \ \left[s^3 + a_1 s^2 \ + \ a_2 s \ + \ a_3\right] \ F(s) \ = \ G(s)$$

Here $G(s)$ is the sum of the Laplace transform of g and a quadratic polynomial involving f(0), f'(0), f "(0) so $F(s)$ is given by

$$F(s) \;=\; \frac{G(s)}{Q(S)}$$

with $Q(s)$ a cubic polynomial in s. F(s) will have singularities at the zeroes of Q and the singularities of G. In particular if G only has poles, then also $F(s)$ only has poles.
A formula for the inverse Laplace transform of a function that only has poles in a half plane is given by the Bromwich integral as follows.

Theorem Suppose $F(s)$ is a function that is analytic in a half plane $Re(s) > s_0$ and only has a finite number of poles in the half-plane $Re(s) < s_0$ and that there are M, R such that

$$|s|^2 |F(s)| \leqslant M \quad \text{when} \quad Re(s) > s_0 \text{ and } \quad |s| \geq R$$

Then the unique function with Laplace transform F(s) is

$$f(t) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds \qquad \text{so}$$

$f(t) =$ sum of residues of $e^{st} F(s)$ in $Re(s) < s_0 < c$.

The zeroes of the polynomial $Q(s)$ are the **resonances** of the system governed by the differential equation.

The usual proof of this inverse Laplace transform formula is based on that for the inverse Fourier transform. Then some properties about change of variables in Lebesgue integrals provide the Bromwich formula given above.
Note that if the original function $f(t)$ is given by a power series

$$f(t) \ := \ \sum_{m=0}^{\infty} a_m \ t^m$$

that converges for all $t > 0$, then the Laplace transform is

$$F(s) \ := \ \sum_{m=0}^{\infty} \frac{a_m \, m!}{s^{m+1}}.$$

If this formula for the Laplace transform is known, then the series for f(t) can be evaluated.