Math 3340: Fixed Income Mathematics
II. Elementary Bond Analysis

For much of this section, suggest that you look at Google or DuckDuckGo or other internet search engines to see descriptions of how bonds are bought and sold from a non-mathematcial viewpoint.

The central point is that prices of bonds are related to the yield, or internal rate of return (IRR). Buyers want higher yields so they prefer smaller discount factor on future bond payouts. In general one expects that

Cost of a bond = Present Value of future payments by the issuer. So we will study formulae for this present value.

Just as there is an APR associated with any discrete interest rate so also there is a continuous interest rate associated with any discrete interest rate. Suppose r_2 is an interest rate per 1/2 year, r_4 is an interest rate per quarter, r_{12} is an interest rate per ordinary month and r_{52} is a weekly interest rate. Then the continuous interest rate r_c associated with these rates satisfies

$$e^{r_c} = (1 + r_2)^2 = (1 + r_4)^4 = (1 + r_{12})^{12} = (1 + r_{52})^{52}$$

Each of these is the 1-year growth factor at the indicated rate.

Exercise What is the similar formule for the 4-week interest rate r_{13} and the daily interest rate r_{365}? Complete the following formulae for r_{13}, r_{52}, r_{365}.

$$r_c = 2 \ln(1 + r_2) = 4 \ln(1 + r_4) = 12 \ln(1 + r_{12}) = ...$$

When a continuous interest rate r is known, then the growth factor of an investment of \$A for time T is

$$f(T) := e^{rT} \qquad \text{so} \quad A(T) = A\, e^{rT}$$

The present value of a payment of \$A to be made at time T in the future is

$$PV = d(T)\, A \quad \text{with} \quad d(T) = e^{-rT} = \frac{1}{f(T)}$$

US Treasury Bills or **T-bills**

See Wikipedia entry for United States Treasury security.

These are issued by the US government who promise to pay you \$1,000 per bill in 4, 8, 13, 26 or 52 weeks time from the date of purchase. An M week bond has a **term** or **time to maturity** of M weeks.

Buyers pays \$10 P per bill with $P < 100$ being determined by an auction. Since these are government bonds they are considered to be “riskless” and have the lowest interest rates of any bonds available in the US for this time period. So T-bill rates are used for comparison purposes to all other interest rates.

Suppose that an M week T -bills is sold for a price of \$10P, then we say that the M-week treasury (continuous) interest rate is r_k per year where

$$P = 100e^{-(k\,r_k)} \text{ and } \quad k = 52/M.$$

Thus the 52-week T-bill rate is the annual continuous interest rate associated with this purchase. The 26-week T-bill rate is the semi-annual continuous interest rate per year The 13-week T-bill rate is the quarterly continuous interest rate and the 4-week T-bill rate is the (lunar) monthly interest rate per year.

A plot of interest rate against time to maturity, or term, is called a **yield curve**.

Example 1. If a 12-month T-bill costs \$980, then we say its price is P = \$98.00. The 52-week T-bill continuous rate is r_a where

$$e^{r_a} = 100/98 = 1.0204082. \quad \text{Thus}$$

$$r_a = \ln 1.0204082 = 0.02020271$$

This is a continuous rate of 2.02% per year.

Example 2. A 6-month T-bill costs \$991.00 so its price is said to be P= \$99.10. The 26-week T-bill continuous rate is r_2 where

$$e^{r_2} = 100/99.1 = 1.0090817 \quad \text{Thus}$$

$$r_2 = 2 \ln 1.0090817 = 0.0180815$$

This is a continuous rate of 1.808% per year.

Zero Coupon Bonds

T- bills are special examples of what are called zero coupon bonds (ZCB). These are bonds that do not make any interest payments but which will pay a given amount \$A at time T from now. A is called the **face value** of this bond and T is the term or the time to maturity. The present value of a bond with face value \$A, at the continuous interest rate r is

$$PV \;=\; A\, e^{-rT}$$

We say the price of such a bond is $P \;=\; 100\, e^{-rT} < 100$ and assume that $r > 0,\;\; T > 0$. So the price is the cost of such a bond with face value \$100 at maturity.

Example. What is the price of a ZCB that pays a continuous interest rate of 3% and has 3 years to maturity?
Answer. $P = 100 \ e^{-0.09} = \$91.3931$. Thus each \$1,000 bond will cost \$913.93.

Problem A 4-year ZCB is bought for \$900. What is the continuous interest rate for this bond?

This time want to find r, when $900 = 1000 \ e^{-4r}$. .
Thus $e^{4r} = 1.111111111$ so $4r = .1053605$ and $r = 0.02634$ or the interest rate is 2.634%.

In this problem, the solution for r is often called the **internal rate of return** (IRR) or the **yield** on this bond. This IRR is then denoted by y (for yield).

Usually one has that the interest rate on shorter term bond from a company or government to be less than that of a longer term bond so one expects the yield curve to be an increasing function of the time-to maturity (or term). If there is a range of "terms" where the yield decreases as the term increases then one has an " inverted" yield curve and a trader could make a profit using riskless "arbitrage" . This could happens when there is likely to be a big change between the maturity dates such as a declaration of war or a difficult election.

Banks usually pay interest rates on savings and CDs that are below the annual interest rates on T-bills - as they often purchase T-bills with the funds. On Wikipedia, and in older texts they describe the "discount yield" of a T-bill. This is a number that is very, very close to this continuous yield per year.

Bond pricing formulae on the internet.

There are many websites on the internet that provide **bond pricing formulae** or descriptions of **bond valuation** They use different termnology and symbols. Most of them give the price for discrete interest rates - not continuous interest rates.

I suggest that you look at the section called Bond Valuation at **investopedia.com.** The Wikipedia article is much more advanced and completely different. Other sites include

wallstreetmojo.com, educba.com, corporatebondfinance.com, xplaind.com, ...

If you find a website that treats continuous interest formulae in a straightforward manner please let me know by email.

All of the web-sites say the price of a bond is based to the **present value of the future payments by the bond issuer.** So it depends on (i) the number and frequency of the **interest payments**. This is called the "interest schedule".

(ii) The **face value** of the bond is the payment a holder receives when the bond matures.

(iii) The **IRR, or yield,** associated with the discount rate on future payments.

All the different formulae should provide approximately the same results for a bond with the same interest payments, face value and discount rate. In this course, we shall mostly treat the formulae for continuous interest rates for bonds with face value $1,000.

Usually interest payments on a bond are described explicitly. A bond that pays \$C every half year is said to have an **original interest rate** $r_O := 2C/1000 = C/500$. 2C is called the **coupon** - because originally buyers of bonds received a (fancy, printed) bond with coupons attached. When an interest payment was due, you cut off the coupon and took it to a bank who would then pay the coupon - as if it was a check for \$C.

When the cost of buying a bond with face value \$1,000 is \$A, then the **price** of the bond is \$P where $P = A/10$. This "price" is the cost of one-tenth of a \$1,000 bond. A bond is said to trade at **par** if its price is P $=$ 100; it is below (above) par when $P < (>)100$.

A bond that has a yield (IRR) y has a (uniform) discount rate $d(t) = e^{-yt}$ for the payment at time t. Suppose the first payment will be τ years from today and there are M payments, then the succesive interest payments are at times

$$\tau, \tau + 1/2, \tau + 1, \tau + 3/2, \ldots, \; t_M = (M + 2\tau - 1)/2$$

The m-th payment is at time $t_m := (m + 2\tau - 1)/2$ years from now, and $0 < \tau \leq 1/2$. These interest payments are an annuity so the formulae we derived earlier hold.

The present value of these interest payments is

$$PVI = C \; e^{-y\tau} \left[\frac{1 - e^{-My/2}}{1 - e^{-y/2}} \right] < M \; C \; e^{-y\,\tau} \leq M \; C$$

from the formulae for annuities given earlier. The discount rate for the m-th payment is $d_m = e^{-y\,t_m}$.

Thus the present value all the payments is

$$PV \;=\; 1000\, e^{-y\, t_M} \;+\; C\; e^{-y\tau} \left[\frac{1 \;-\; e^{-My/2}}{1 \;-\; e^{-y/2}} \right]$$

where the first term is the present value of the payment at redemption. Thus the price of this bond with an IRR y is

$$P(y) \;:=\; 100\, e^{-y\, t_M} \;+\; \left(\frac{C}{10\; e^{y\tau}} \right) \left[\frac{1 \;-\; e^{-My/2}}{1 \;-\; e^{-y/2}} \right]$$

This is usually written as the **price-yield formula** for a bond,

$$\frac{P(y)\, e^{y\tau}}{100} \;=\; e^{-My/2} \;+\; \frac{C}{1000(1 \;-\; e^{-y/2})} \left[1 \;-\; e^{-My/2} \right]$$

Here the first term on the RHS is due to the maturity of the bond while the second is associated with the interest payments.

For y near 0, one can show that

$$f(y) \ := \ \frac{1}{(1 \ - \ e^{-y/2})} \ = \ \frac{2}{y} \ + \ \frac{1}{2} \ + \ c_2 y \ + \ \dots$$

by using Taylor's approximation for the function $yf(y)$ near $y = 0$. If one just uses the first term this leads to the **approximate bond pricing formula**

$$\frac{P(y)\, e^{y\tau}}{100} \ = \ e^{-My/2} \ + \ \frac{C}{500y} \ (1 \ - \ e^{-My/2})$$

When the second term is included one finds the better approximation

$$\frac{P(y)\, e^{y\tau}}{100} \ = \ e^{-My/2} \ + \ \frac{C\,(1 \ + \ y/4)}{500y} \ (1 \ - \ e^{-My/2})$$

Another version of the price yield formula is that

$$\frac{P(y)\, e^{y\tau}}{100} \;=\; \left[1 \;-\; \frac{C\, f(y)}{1000}\right]\; e^{-My/2} \;+\; \frac{C\, f(y)}{1000}$$

where $f(y) \;:=\; (1 \;-\; e^{-y/2})^{-1}$ is the function introduced above. The two approximations just involve different expressions for the function $f(y)$ in this formula.

The formula that is usually given on websites and older texts, requires that $\tau = 0$ and then

$$\frac{P(y)}{100} = (1 + y)^{-M/2} + \frac{C}{500y} \left[1 - (1+y)^{-M/2}\right]$$

where y corresponds to the 6-month (half-year) interest rate.

For comparison with the formula with continuous interest rates, this is

$$\frac{P(y)}{100} = \left[1 - \frac{C}{500y}\right] (1 + y)^{-M/2} + \frac{C}{500y}.$$

They are obtained by using the annuity formula with payments every half-year starting in 6 months time - so it works when a bond is being sold initially. A number of different adjustments have been used when the time to the first payment is significantly less than 6 months.

Properties of the Bond Pricing formulae

All of these approximations have expressions on the RHS that are functions of C, f(y) and M. By taking partial derivatives, you can show

1. the bond price is linear (affine) and increasing in C.
2. the bond price is a decreasing, convex function of the yield y,
3. the bond price is an increasing function of M.

These properties hold for both the exact formulae and the approximations and are intuitively natural. Think about what what they imply, such as the higher the interest payments, or the more payments, the higher the price. In particular a bond that pays interest will always cost more than a ZCB with the same yield and time to maturity.

Example What is the cost of a 5 year bond that has a coupon of \$12 each half-year and 5 years to maturity with the first payment in 6 months time if it is priced for an IRR of 1%?

Note that the interest payments imply the original interest rate on this bond is $0.024 = 2.4\%$. So the original rate is higher that the IRR of $y = 0.01$. The continuous bond pricing formula gives

$$\frac{P\, e^{-y/2}}{100} \;=\; e^{-5y} \;+\; 0.012\, \frac{(1 \;-\; e^{-5y})}{(1 \;-\; e^{-y/2})}$$

upon substituting for M, C and τ. Thus when $y = 0.01$, one finds $P = 108.407$. The actual cost of a \$1000 bond then is \$1084.07. This bond costs more than \$1,000 because it pays interest at a higher rate than the IRR. Note that whenever the coupon is more than \$5 every 6 months, and current interest rates are about 1%, one should expect to pay above par for a bond.

The above calculation used the exact formulae for this continuous yield. If you use any of the approximate formulae or the discrete formula the answers should be quite close to this answer.

In Kellison's text (which used to be recommended for actuarial exams), chapter 6, he describes 3 different "yields" on page 202 and gives 4 different bond pricing formulae. He calls this IRR, the yield to maturity and the original yield r_O is the nominal yield.

The **current yield** of a bond with coupon $C each half year bought at a price P is

$$y_C \ := \ \frac{C}{5P}.$$

This is the ratio of the annual interest payments ($2C) to the cost of buying the bond ($ 10P). It does not depend on the time to maturity or number of payments

The **capital gains yield** of a bond ignores the interest payments and just depends on the price paid (P) and the time to maturity (T). It is the solution y_{cg} of

$$P\ e^{yT} \ = \ 100 \qquad \text{so} \qquad y_{cg} \ := \ \frac{1}{T}\ \ln\left(\frac{100}{P}\right)$$

You see that this yield is positive when $P < 100$ and is negative if $P > 100$.

The total yield to maturity then is

$$y_T \ := \ y_C \ + \ y_{cg} \ = \ \frac{C}{5P} \ + \ \frac{1}{T}\ \ln\left(\frac{100}{P}\right)$$

It does not make sense for an investor, bank or insurance company to buy a bond whose yield to maturity is negative. This means that they are paying more for the bond than they will receive back.

The total yield, is a decreasing function of P, and it will be positive when

$$\frac{C}{5P} + \frac{1}{T} \ln\left(\frac{100}{P}\right) \geq 0.$$

Let $x = 100/P$ and $\hat{x}$ in (0,1) be the solution of the equation

$$\ln x = -\frac{CTx}{500}$$

then the value $P_{max} := 100/\hat{x}$ will be the maximum price for a bond with coupon C and time T to maturity.

Example. A 3% bond has 2 years to maturity and is priced at 99. What is its total yield to maturity?

Ans. The current yield is $y_C = 3/99$ and the capital gains yield is $2y_{cg} = \ln(1.01010101)$. so $y_{cg} = 0.005025, y_C = 0.030303$, and $y_T = .035328$ or 3.53%. It is more than 3% because the price is below par and there is a copital gains. If you paid \$1,010 for the bond then P = 101 and you would have a capital loss so the total yield would be below 3%.

This is a nonlinear equation that is often written as

$$x \;=\; \exp\left(-\alpha x\right) \quad \text{with} \quad \alpha \;=\; \frac{CT}{500}$$

It has a unique solution since this LHS is a straight line increasing from 0 to 1 as x goes from 0 to 1. The RHS is a function that decreases from 1 at x=0 to $e^{-\alpha} < 1$ at x=1. These two functions will cross at the unique point $\hat{x}$ between 0 and 1.

Example. A bond with 5 years to maturity and a coupon of $12 will have a maximum price of $P_{max} \;:=\; 100/\hat{x}$ where $\hat{x}$ is the solution of $x \;=\; \exp\left(-\alpha x\right)$ with $\alpha = 0.12$.

Return on Equity from an Investment

Very often people borrow some of the funds used to make an investment. Suppose that an investment has an initial cost (= value) of $\$V_0$ which consists of E_0 of your own funds and $\$B_0$ that you borrow at continuous interest rate r.

At a later time t, the value of the investment is $V(t)$. If you have not made any payments on the loan, then your outstanding balance will be $B_0 e^{r\,t}$. The difference between the value of the investment and your debt is called your **equity** in the investment and is given by

$$E(t) \;:=\; V(t) \;-\; B_0 e^{r\,t}$$

The **rate of return on equity** on this investment at time t is given by r_E where

$$E(t) \;=\; E_0 e^{r_E\,t} \qquad \text{so} \quad r_E(t) \;:=\; \frac{1}{t}\,\ln\left(\frac{E(t)}{E_0}\right)$$

Example You buy 200 shares of CCI at \$150, with \$18,000 of your own funds and \$12,000 lent by your broker at 6% interest compounded continuously and allowed to accumulate in the account. A year later you sell the shares for \$170 and the account is closed. What was your profit and your return on equity? Would you have done better by just buying 120 shares and not borrowed on margin?

Answer. Here $t = 1, r = 0.06, E_0 = 18000, B_0 = 12000$ and $V(1) = 34,000$. Thus, after 1 year,
$E(1) = 34000 - 12742.04 = 21,257.96$. So your profit was
$E(1) - 18000 = 3257.96$ with $E(1)/E_0 = 1.181$ and
$r_E(1) = 0.16136$ or 16.14%.

If you had not borrowed any funds, your 120 shares would be worth \$20,400 so the profit would have been \$2,400 and the continuous 1-year rate of return would have been $r_1 = \ln(17/15) = .12516.$ This says that the shares provided a 12.52% annual rate of return.

You increased your profit and the rate of return by using the margin loan from your broker, since the rate of return on the shares is greater than the interest rate charged by the broker.

If the increase in the value of the asset, is less than the interest rate on the loan then your return on equity will decrease. Investors need a formula for determining the return on equity as a function of the amount borrowed to decide how much, if any they should borrow for a project.

Here we'll assume that T is known As an investor in this arrangement, your primary concern is with the gain or loss you make on your investment of E_0 This is measured by the **rate of return on equity** which is

$$r_E(T) := \frac{1}{T} \ln \frac{E(T)}{E_0} = \frac{1}{T} [\ln E(T) - \ln E_0]$$

It turns out that the formula for $r_E(T)$ depends on the rate of return on the investment, the interest rate on your loan and a quantity called the **debt to equity ratio**. It is usually denoted $\delta := B_0/E_0$.

In finance one usually are most on how $r_E(T)$ depends on δ - since this is a choice you have at the beginning.

The formula for the return on equity is that $r_E(T) := \hat{x}$ where x is the solution of the equation

$$e^{x\,T} = (1 + \delta)\, e^{r\,T} - \delta\, e^{r_b\,T}.$$

Here r is the rate of return on the investment and r_b is the interest rate on the loan.

Proof: The value at time T of the investment is $V(T) = e^{rT}\,(B_0 + E_0)$. It also $= e^{r_E\,T}\,E_0 + e^{r_b\,T} B_0$ from the definition of equity. Divide both sides by E_0 and you obtain the descired equation.

This formula may be written in terms of annual interest rates (APRs) as

$$(1 + r_E)^T = (1 + \delta)\,(1 + r)^T - \delta\,(1 + r_b)^T$$

When $T = 1$ the return on equity after 1 year is given by

$$r_E = r + \delta (r - r_b)$$

Note that,in both versions of this formula and for fixed T, the value of r_E in this equation increases as r increases. If $r < r_b$, then the RHS is less than $e^{r\,T}$ so that $r_E < r$. while if $r > r_b$ then the RHS is an increasing function of δ or your return on equity will increase as δ increases.

Those who understand compound interest, earn it,
those who don't pay it.

Payments for a Lease

Another common financial arrangement that can be modelled as an annuity is a **lease.** A simple lease such as for furniture, a car, or a piece of equipment is an annuity where the buyer makes an initial payment, a fixed number of monthly payments and the lender recovers the equipment at the end and values at a final amount.

So the lessor owns equipment with initial value $\$V_0$, final value $\$V_F$ while the leasee has the use of the equipment for an initial payment of $\$P_0$ and M monthly payments of $P.

The leesor needs to estimate the final value and decides on a discount factor for the lease. For a lease of a Ferrari, the final value may be much less than the initial value, while with real estate leases the final value may well be close to the initial value.

To evaluate a lease a lessor will balance the change in value of the asset over the M months by the payments made by the leasee. She views the cost of providing the property as $V_0 \; - \; d^M V_F$ where d is her monthly discount factor. The present value of the leasee's payments are $P_0 \; + \; (d + d^2 + ... + d^M) \; P$. Equating these leads to the **lease payment equation**

$$V_0 \; - \; P_0 \; - \; d^M V_F \; = \; \frac{d\,(1 - d^M)\,P}{1 - d} \; = \; v_{M1}(d) \; P$$

where v_{M1} is the function used in the theory of annuities. This is a simple equation for P.

This formula also applies when the lease payments are made on some other regular basis - such as weekly or quarterly. In such cases the discount factor should be for the same time period as the payments.

Example. A real agent leases a new BMW with an initial value of \$66,000 for 3 years. The dealer requires a initial payment of \$6,000 and expects the vehicle will have a residual value of \$40,000 after 3 years. If the dealer prices the lease at a monthly discount factor of $d = 0.99$, what is the monthly payment on this lease?

Note that this discount rate corresponds to an APR of 12.82% - so she is charging quite a lot for the lease. Still there are 36 payments that need to cover a decrease in value (after the deposit) of value \$20,000 for the Bimmer. So the cost of the lease will be at least \$556 per month.

Answer. With these numbers the lease payment equation becomes

$$(0.99)(0.303587)P \; = \; (0.01)[60000 - (0.696413).40000] = 32143.47$$

so $P = \$1,069.48$ per month. Suggest that you look at the cost of a lease when the discount factor is only $d = 0.995$ (say).

Sometimes the formula is written

$$v_{M1}(d)\; P \;+\; d^M\; V_F \;=\; V_0 - P_0$$

and the left hand side is linear in P and V_F and you can ask how does the monthly payment depend on the residual value V_F, or the initial payment P_0? Obviously the larger the value of V_F the lower the monthly payments for the lease - assuming d remains the same.

Duration of an Annuity or Bond

Usually a person buying an annuity is interested in both how much they will receive from the issuer and when they will receive it.

If it is a ZCB, then the answer for when is simple - at maturity. For a more general annuity, the usual measure of when you receive the payments is some average time of the payments. If the annuity has payments A_m at times t_m with $1 \leq m \leq M$ and \$A is the total amount received then a simple measure is

$$\overline{T} \ := \ \frac{1}{A} \sum_{m=1}^{M} t_m \, A_m \quad \text{with } A \ = \ \sum_{m=1}^{M} A_m$$

This quantity does not involve the time value of money, so the average that is used by most lenders is the **Macaulay duration** of the annuity. This is

$$T_{Mac} := \frac{1}{PV} \sum_{m=1}^{M} t_m \, PV_m \quad \text{with } PV = \sum_{m=1}^{M} PV_m$$

where PV_m is the present value of the m-th payment and PV is the present value of all the payments (and is close to the cost of the annuity). Observe that $t_1 \leq T_{Mac} \leq t_M$ and this is a **weighted average** of the present values of the payements. T_{Mac} is usually measured in years.

T_{Mac} is often considered to be the time required for the annuitant to receive back the cost of the annuity. It is always less than t_M and equals t_M only when there is just a single payment.

It is used by buyers who need to hold bonds that will cover payments at specific times in the future. They want to own bonds with specific Macaulay duration

There also is a different quantity called the **modified duration** of an annuity. Sometimes it is called the **volatility** or **yield volatility** of the bond.

For a bond with IRR y, the modified duration is the quantity

$$\nu(y) \; := \; -\frac{P'(y)}{P(y)}$$

and has units of time. When y is an annual rate of return (either continuous or APR) then $\nu(y)$ has units of years. It measures the rate of change of the price as interest rates change. For a ZCB with T years to maturity and an annual IRR of y, one has

$$P(y) \; = \; 100/(1+y)^T \qquad \text{so} \quad P'(y) \; = \; \frac{-100\,T}{(1+y)^{T+1}}$$

Thus $\quad \nu(y) \; = \; T/(1+y) \quad$ while $\; T_{Mac} = T.$

Wikipedia has a article on duration that has a long comparison of Macaulay duration and modified duration. They both have units of time but measure different aspects of a bond. The above calculation for ZCBs shows that they provide similar, but different, values when y is small.

Bond investors are usually very concerned by how the price of the bonds will change when interest rates change. Often they make more money from capital gains than from the interest payments. Remember from the price-yield formula formula for bonds, the price goes down when the yield goes up so $\nu(y)$ always is positive.

When $\nu(y)$ is known, then $P(y)$ can be found using elementary theory of differential equations.

Example. An investor in November 2015 bought a 10 year zero coupon bond at a price of \$72. Now in 2020 he can sell the bond for \$94. What was his rate of return on this bond investment and how does it compare with the IRR at time of purchase?

Answer He paid \$720 for a bond and sold it 5 years later for \$940, so the annual growth factor is f where $940 = 720f^5$. So $f = 1.054773$ or he had a 5.478% annual gain in value.

If he had held the bond to maturity, the IRR would have been f_m where $1000 = 720f_m^{10}$. Thus he bought it with an IRR of $f_m = 1.033396$. Its current IRR is f_c where $1000 = 940f_c^5$ so $f_c = 1.01245$, or the buyer will obtain an IRR of 1.245% if they hold it to maturity.

So he bought the bond with an expected annual yield to maturity (ytm) of 3.334%. When he sold the bond, he had actually made a annual return of 5.48% on the investment and the buyer is only expecting an annual ytm of 1.245%. The buyer may earn a higher return if interest rates continue to fall. If they rise, they should keep the bond in his portfolio - unless they have to sell for other reasons.

Thus bond traders are very interested in whether interest rates are going to go up or down as they wil cause bond prices to go down or up respectively.

The price formula derived for a general annuity depends on the payments and a uniform discount factor $d := 1/(1+y)$. When the cost is $C(d)$, the duration of the annuity will be

$$\nu(d) := \frac{C'(d)}{d^2 \ C(d)}$$

where $C'(d)$ is the derivative of $C(d)$. It measures how much the cost changes as the discount factor changes. In particular the cost of an annuities should increase as discount factors increase.

Convexity of Bond Prices

The convexity of a bond price is proportional to the second derivative

$$c(y) \ := \ \frac{P''(y)}{P(y)}$$

For a ZCB, this is given by

$$c(y) \ = \ \frac{T\,(T+1)}{(1+y)^2}$$

In general this function is a positive function that is a decreasing function of y. Usually it is also an increasing function of time to maturity - or durations.

The Yield Curve

The preceding material has described the prices of bonds as depending on an **internal rate of return,** or **yield;** usually denoted by y.

This variable "y" is central to everything about bond pricing just as the number R_0 is the main quantity that every epidemiologist uses to describe how a virus such as C-19 spreads.

It wasn't until the 1970's that people were able to study historical data sufficiently well that they could propose some theories about what determines this variable for use in these formulae.

The yield curve is constantly changing and is used by observers for predicting possible economic and financial events. The US Treasury department has web pages, updated daily, that provide **US yield curve rates** based on their operations. They regularly buy or sell billions of dollars of many different types of bonds a day.

The yield curve is a graph of the implied yield on US government zero coupon bills or bonds, as a function of time to maturity (ttm). From the prices paid for these bonds, their yield can be determined - as in a mid-term exam question. You may find a good description of some features about yield curves at

https://xplaind.com/128027/yield-curve

Alternatively see the Wikipedia or Investopedia articles on "yield curve."

There are at least three major theories about what determines these curves and most finance professionals agree that this is one of the major open questions / problems in finance and economics.

For this class the main thing you need to know is that there are such curves, they are what is actually used for determining the prices of bonds in various markets. When we use a yield curve, we will generally assume it is an increasing function of ttm that becomes almost constant after 10 years.

People who are recognized experts on the yield curve can earn salaries of more than a million dollars a year - or do very well as consultants or advisors.

Arbitrage of Bonds

Suppose that a bond trader can buy a 1 year Treasury bill for \$99.01 a 2-year US ZCB bond \$97.20 and a three year US ZCB for \$95.67.

Since the price of a ZCB with time T to maturity and IRR y is

$$P(y, T) = \frac{100}{(1+y)^T}$$

the yield is

$$y = \left(\frac{100}{P}\right)^{1/T} - 1$$

Hence the yields on these bonds are

1%, 1.43%, 1.486% respectively.

At finance.yahoo.com search for "Treasury bond rates" to see today's values.

A bond trader sees that he can buy a 3-year US government bond that pays \$30 interest once a year for \$P, so he buys 100 of them. At the same time, he "sells" to a university 3 1-year treasury ZCBs at 99.01, 3 2-year ZCB at 97.20 and 103 3-year bonds at 95.67. That is he promises to pay the university \$3,000 in a years time, \$3,000 in 2-years time and \$103,000 in 3-years time.

The university pays the current market price for these bonds which is

$$30 \times (99.01 + 97.20) \; + \; 1030 * 95.67 \; = \; 104,426.40.$$

If his firm can buy the 3-year bonds for any amount less than this, they will have "covered" their promise to the university.

Suppose he buys the 2% bonds for P=103 each. His cost is $103,000, for the 100 3% bonds. When the US treasury makes its interest payments, they are forwarded directly to the university. His firm banks an immediate gain of $1,426.40 on this transaction - and minimal further financial concerns about the transaction. These are funds that “settle” immediately and solely involve US government payments - so they are regarded as “riskless”.

This is the basis of **arbitrage** where a trader simultaneously buys and sells contracts with the same payouts - making a (preferably low-risk) profit in the process. These profits are due to “market discrepancies” where certain bonds are “mispriced” compared to others. It is **risk-free** in that his profit “certain” and immediate.

In financial theory, people assume that prices are based on a no-arbitrage principle because if there is an arbitrage possibility someone will take it.

Given this yield curve, traders view the correct price for a 3% US treasury bond with 3 years to maturity to be about $104.63. If you could buy such a bond for less than this amount, do so and make an immediate profit by selling it to any buyer that is looking to receive these payments. That buyer should not buy this bond if its price is higher than $104.63 because they could receive the same income stream by buying the ZCB's directly at today's prices. So in an **efficient market** the price of the bond would be exactly $104.63. This is called **pricing with respect to a yield curve.**

To summarize; **The prices** of all other US government bonds are determined by the prices of federal government ZCB's that have the same **payment schedule** as the bond itself.

In mathematics a theorem usually has the form "If ... holds, then ... is true (or false)". In financial trading you often have situations where there are two annuities X and Y and "The price of X and Y should be the same because they will provide the same payments (returns) in the future." However you can buy X for P_x and sell Y for P_y with $P_x < P_y$.

So traders look for situations where prices "should" be the same but are not. Then they look to see if they can make a profit by buying one and selling the other.

Wikipedia has a good, somewhat technical article on **arbitrage** that includes some sections on the arbitrage of bonds. In particular they give 3 conditions that will enable successful arbitrage. It is one of the first topics in **financial engineering** and is fundamental for a tremendous amount of financial trading.

Much more about this can be found in the text "Investment Science" by David Luenberger that covers much more about the models and mathematics involved.

Spot Rates and Forward Rates

The interest rates in the yield curve are often called **spot rates.** They are the interest rates that hold today for ZCB's with T years to maturity. They may be based either on discrete interest rates or continuous interest rates. We will just describe the continuous case here.

Suppose someone wants to arrange to borrow funds at a time T_1 in the future with repayment at time $T_2 > T_1$ years in the future. The interest rate on such a ZCB, fixed today, is called the **forward rate** f_{12}. See the article on "Forward Interest Rate" in Wikipedia.

In recent years, financial trades are not based on the yield curve for annual IRR but on that of **spot rates** The spot rate for a time T is the continuous yield of a zero-coupon US government bond that matures T years from now. This is the solution of

$$P(T) \; = \; 100e^{-s\,T} \qquad \text{or} \quad s(T) \; := \; \frac{1}{T} \, \ln\left(\frac{100}{P(T)}\right)$$

where $P(T)$ is the price of the ZCB with ttm T.

The spot rates for the ZCB examples given above are

$$s(1) \; := \; .00995, \quad s(2) \; := \; 0.01420, \quad s(3) \; := \; 0.014756$$

Note that these all satisfy $s(T) < y(T)$ since continuous rates are less than annual rates.

The graph of the function $s(T)$ against T is called the spot-yield curve and T is measured in years. The number $s(T)$ is usually given as a decimal so in the US it has been in the interval $(0, 0.018)$ for the last 100 years. Suppose that the spot rate for a time T_j is denoted s_j.

The forward rate from time T_j to T_k with $0 \leq T_j < T_k$ is the number f_{jk} which will be the interest rate that you would pay now to obtain a ZCB at time T_j that will mature at tme T_k. That is the requirement that buying a ZCB bond now to mature at time T_k, should be at the same price as buying a bond now to expire at time T_j and then buying the "forward contract"

The cost of ZCB's bought now to mature at times T_j, T_k will be

$$P(T_j) \ := \ 100 \ e^{-s_j \, T_j}, \qquad P(T_k) \ := \ 100 \ e^{-s_k \, T_k}$$

so the forward rate f_{jk} is the value such that

$$e^{-s_k \, T_k} \ = \ e^{-f_{jk} \, (T_k - T_j)} \ e^{-s_j \, T_j}$$

That is

$$f_{jk} \ := \ \frac{s_k \, T_k \ - \ s_j \, T_j}{T_k \ - \ T_j}$$

Example. What is the forward rate to buy a contract for a 2 year ZCB in one year's time with the spot rates given above?
Answer Want to find the value f_{13}. Substituting in the formula

$$f_{13} \ = \ [3 \, s(3) - s(1)]/2 \ = \ .017159$$

This is useful to everyone interested in borrowing or lending money as it is a **prediction** of what the 2-year spot rate will be in 1 year's time.

A person who will receive a million dollars in a year's time and will then need it again in 3 year's time is able to ensure that it will earn interest at a continuous rate of 0.01716 for those 2 years by buying a forward contract at this rate. The interest rate can be fixed now - you don't have to wait to see what the spot price will be in November 2021!

So if you think the wrong party will win the election next week and interest rates will go down a lot you should buy a forward contract. If you think that interest rates will go up, just wait till they do and you'll do better.

The No-Arbitrage Inequality

There are other important consequences of these formulae. Suppose that the 1-year spot rate is $s(1) = .03$. Then the cost of a 1-year ZCB will be $P(1) \; = \; 100e^{-s(1)} = 97.045$

So a buyer could buy a 1-year bond, keep the proceeds under their bed, or in a non-interest bearing account, and still have \$1,000 after two years. This costs less than buying at the current two year spot rate of 97.20.

Similarly if the 3-year bond rate is $s(3) = 0.009$ the cost of buying a 3-year ZCB would be $P(3) \; = \; 100\, e^{-.027} \; = \; 97.34$. In this case it would be better to buy a 2-year ZCB at 97.20 and hold the payout in a non-interest bearing account for the last year!

So, when a spot rate at time T is known, then the probability of arbitrage implies that there will be bounds on what the spot rates could be either before and after T. These are the **no-arbitrage inequaities** for spot-rates.

Theorem (no-arbitrage) The function $T\,s(T)$ is an increasing function of T.

This implies that if $T\,s(T)$ is known for some value $T > 0$, then

(i) $0 < s(t) < (T\,s(T))/t$ when $0 < t < T$.

(ii) $s(t) > (T\,s(T))/t$ when $t > T$, and

(iii) $f_{jk} \geq 0$ whenever $0 \leq T_j < T_k$.

Note that if the function $s(T)$ is an increasing function of T so is $Ts(T)$. However a function that satisfies this theorem need not be an increasing function everywhere. Time intervals during which $s(t)$ is a decreasing function of t are often observed - and the yield curve is then said to be inverted. However

Example. Find upper and lower bounds on the spot rates for $t < 2$ and $t > 2$ from the preceding value of $s(2) = 0.0142$.
Answer When $0 < t < 2$, then (i) says that $0 < s(t) < .0284\, t^{-1}$.

Similarly (ii) says that if $t > 2$, then $s(t) > 0.284/t$.

For $0 < t < 1$ one see that for this data the no-arbitrage inequality implies

$$s(t) < 0.00995\ t^{-1}, \quad \text{and} \quad s(t) < 0.044268\ t^{-1}$$

from the data for T = 1,3 respectively. Thus the value from T=1, gives the best upper bound.