Math 3340: Fixed Income Mathematics or The Theory of Interest Fall 2020

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A prerequisites and course description are available on my web site. A syllabus with lots of official information is now available on the Access UH website

This course will not follow any current text very closely.

The theory of interest has been studied since biblical times in many different parts of the world. It is essential for all trade and business as **merchants** (people who buy and sell goods) are in the situation where they pay for the goods at one time and then receive payment from someone else later.

Bankers, pawnbrokers, mortgage companies, factors and many different types of lenders make their living by arranging transactions that include interest charges. Interest is the **cost of borrowing, or lending, money.** The mathematical theory of interest is the study of the formulae involved.

Will and Ariel Durant in "The Lessons of History" said

... those who manage money manage all.

For a long time this was a topic where there were few laws. The Truth in Lending Act was only passed in 1968 and provided some legal definitions that will be used here - including APR

This course is about the formulae used for managing money - assuming that everyone is honest and pays. In practice there always is a possibility of **default** or someone not paying what the owe. This means that you must add probability theory to the calculations. So one of the first required actuarial exams is on **probability theory + interest theory.**

The essential results that you should learn from this course include

- the formulae for loans and mortgages.
- How to evaluate the present value of a string of payments - such as pensions and annuities.
- How to price treasury bills and government bonds. The evaluation of current yield, duration, convexity.
- How leverage affects returns on bonds and other investments.
- Formulae for the evaluation of portfolios.

Probability theory is not a prerequisite for this course and will not be used, or needed, here. But to apply the ideas to real world situations, you do need to include probabilistic ideas.

Unfortunately there is no text that describes the simple mathematics behind each of these topics. Hence no required text for the course. When I first taught this course I used the book The theory of Interest, 3rd ed, Stephen G. Kellison, McGraw Hill

More recently An Introduction to the Mathematics of Money, Lovelock, Mendel and Wright, Springer, has been used as a text.

There are hundreds of thousands (sometimes millions) of postings on the Web about each of these topics. Wikipedia has lots of articles on various aspects. In general I'll try to use actuarial notation (as in Kellison's text) but some of that notation is rarely used now and everybody now uses spreadsheets for most of their calculations.

Neither of the above texts uses spreadsheets, or personal computers. They are outdated for teaching - but the theory hasn't changed. It is also hard to use more than 1 source for this material as everyone seems to have a different notation.

You will need to use spreadsheets for doing the homework. If you don't already have one that you know how to use; I suggest Apache OpenOffice which is free and can be downloaded in versions for most operating systems and languages.

A lot of the theory here uses the same mathematical ideas as is used in biology for describing growth and population dynamics. Plants and critters breed rather than earn interest - but the formulae for their population and size are similar and have similar proofs.

Growth Factors and Interest rates

Throughout this course we will be using the symbol A_m for an amount at time t_m and A_m is usually measured in units of US dollars, (sometimes thousands or millions of US dollars.)

At increasing times $\{t_0 < t_1 < t_2 < \ldots < t_m < \ldots\}$ suppose the value of an account is $\{0 < A_0, A_1, A_2, \ldots, A_m, \ldots\}$. Then the ratio

$$f_m \ := \ \frac{A_m}{A_{m-1}} \quad \text{is called the m-th} \quad \textit{growth factor}$$

of the account and measures the change between times t_{m-1}, t_m. Equivalently the value of the account satisfies the equation

$$A_m \ = \ f_m \ A_{m-1} \quad \text{for } m = 1, 2, 3, \ldots.$$

The **interest rate** in the m-th time interval is r_m where $f_m := 1 + r_m$. Note $r_m := f_m - 1$ usually is a small number - and hopefully positive.

Example 1. A person puts \$1,000 in a bank account that pays interest of 1% a month for 12 months. How much does she have after each month and at the end of a year?

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At each time t_m, an interest payment of $\$A_{m-1}/100$ for the period (t_{m-1}, t_m) is added to the account so

$$A_m = 1.01\, A_{m-1} \quad \text{with } A_0 = 1000.$$

That is she receives \$10 after 1 month, \$10.10 (or 1% of \$1010) for interest in the 2nd month etc.

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The growth factor for this account is $f = 1.01$ and the interest rate is $r = .01 = 1\%$ and do not depend on m.

Note that both growth factors and interest rates are simple numbers. A percentage $x\%$ is just the number $x/100$. For example 5% is the number $.05 = 5/100$.

If an account has a uniform interest rate $f = 1 + r$ for M time periods, then the value of an account at time t_m when it started with initial amount A_0 has

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$$A_1 \; = \; f\,A_0, \; A_2 \; = \; f^2\,A_0, \; A_3 \; = \; f^3\,A_0, \; A_4 \; = \; f^4\,A_0, \; \ldots$$

That is

$$A_m \; = \; f^m \, A_0 \; = \; (1+r)^m \, A_0 \tag{1}$$

This is the **Compound Interest Formula**.

In finance we usually just determine amounts at discrete times, such as daily, weekly, monthly, every quarter, half-year or year.

Then the growth factors and interest rates are per day, per week, per month, ... You may think that an interest rate of 1% per month, would be the same as 3% per quarter, 6% per half-year or 12% per year - but they are NOT.

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As an exercise evaluate the value after 1 year of the account that had an initial amount $1,000 and had each of these compounding rates. A calculator yields that $(1.01)^{12} = 1.126825$, $(1.03)^4 = 1.12551$, $(1.06)^2 = 1.1236$ so after 1 year she would have respectively

$$\$1126.82, \quad \$1125.51, \quad \$1,123.60 \text{ and } \quad \$1120.00$$

The difference here could be up to \$7 or so depending on how often the interest is compounded. The more often a given **nominal annual rate** of interest r_a is compounded, the larger the interest amount.

When you earn interest at an annual rate of r_a compounded k times a year, then the amount is given by

$$A_m = \left(1 + \frac{r_a}{k}\right) A_{m-1} \qquad \text{so} \quad A_m = \left(1 + \frac{r_a}{k}\right)^m A_0$$

is the value after m time periods (or m payments).

This is why banks and brokerages usually charge daily interest rates on loans, mortgages usually have monthly interest payments while bonds usually pay interest only every 6 months.

The reason that they are different follows from the binomial theorem - which you should have seen in high school algebra or Wikipedia. It says

$$(1+r)^2 = 1 + 2r + r^2, \qquad (1+r)^4 = 1 + 4r + 6r^2 + 4r^3 + r^4$$

$$(1 + r)^{12} = 1 + 12r + 66r^2 + 220r^3 + \dots$$

There will be 13 terms in this last expression and in general $(1 + r)^m$ has (m+1) terms.

Exercise Find the formulae for $(1 + r)^3, (1 + r)^5$ and the general $(1 + r)^m$. The coefficients here may be evaluated using **Pascal's triangle.** What is that?

Since the actual growth depends on how often interest is compounded, the federal law now requires that whenever an interest rate is given, and in additon to what ever **nominal formula** is described, the **annual percentage rate APR** must be given. The APR of an account is $\hat{r} := \hat{f} - 1$ where $\hat{f}$ is the growth rate on the account over 1 year or

$$\hat{f} := A_{1year}/A_0$$

This enables a person to compare the cost of different interest formulae. Two sets of compounding formulae are said to be equivalent if they yield the same APR and annual growth rate.

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This enables a person to compare the cost of different interest formulae. Two sets of compounding formulae are said to be equivalent if they yield the same APR and annual growth rate. In the preceding example the APRs are, respectively,

$$0.1268 = 12.68\%, \quad 0.1255 = 12.55\%, \quad 12.36\% \text{ and } \quad 12\%$$

depending on whether they are compounded monthly, quarterly, half-yeraly or annually.

Exercise. What is the APR of a nominal interest rate of $0.12 = 12\%$ compounded daily? Find the various interest rates of any loans or credit cards that you, or your family, maintain. See what different rates they describe and see if you can verify their equivalence.

The **Compound Interest Formula** is the basic formula for doing any calculation in interest theory.

$$A_m \;=\; f^m \, A_0 \;=\; (1+r)^m \, A_0 \tag{2}$$

Note that it involves 4 variables A_0, A_m, r and m. It is an **equation**, so whenever three of these numbers are given, there are ways to find the possible value, or values, of the fourth.

Problem 1. The simplest calculation is given A_0, m, r, to find A_m. This problem was solved earlier.

Problem 2. A variation on Problem 1 is to ask how much you need to invest in this account if you want to have an amount \$1,000 after 1 year? That is, given $m = 12, A_12 = 1000, r = 0.01$ to find A_0.

In this case the equation becomes

$$1000 \ = \ 1.126825 \ A_0 \quad \text{since} \quad 1.01^{12} = 1.126825.$$

so $\quad A_0 \ = \ \$887.45$

The general solution is $\quad A_0 \ = \ (1 \ + \ r)^{-m} \ A_m$

This is the expression for the initial amount as a function of the final amount, the nominal rate and the number of payments.

Problem 3. Another question that you can ask is given an interest rate, how many payments are needed before the account doubles in value?

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Take natural logarithms (to base e) of both sides, then

$$m \ln(1+r) = \ln 2 \qquad \text{or} \qquad m = \frac{0.693147}{\ln(1+r)}$$

This is the formula for the **doubling time.** From Taylor's theorem for the logathmic function and $|r| < 1$, one has

$$\ln(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \ldots .$$

When r=.01, the values of the Taylor approximation using 1,2,3 terms and the exact value are

$$0.01, \quad 0.00995, \quad 0.00995033, \quad 0.009950331$$

so with 3 terms of the Taylor approximation you have the exact value to 8 decimal places. In fact a simple approximation for the doubling time is that

$$m = \frac{0.7}{r}$$

Example When the interest rate is $r = .01$ per time period, the exact and the approximate answers for a doubling time M are

$$M = \frac{0.693147}{\ln(1.01)} = 69.66 \text{ and } \quad \tilde{M} = \frac{0.7}{.01} = 70.$$

That is it takes 70 payments for the initial investment to double. When the payments are made monthly this is about 5.8 years or 5 years and 10 months.

Problem 4. The final problem is simple if you have a calculator, but hard if not. Namely if you are given the initial and final amounts and the number of interest payments, what is the interest rate? When m=1 or 2, this can be solved using simple algebra. (Find the answers). When $m \geq 5$, the answer requires calculus - not algebra.

Suppose that a person put \$1,000 into an account that paid monthly interest and after 12 payments had \$1,120. What was the nominal monthly interest rate?

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Suppose that a person put $1,000 into an account that paid monthly interest and after 12 payments had $1,120. What was the nominal monthly interest rate? In this case m=12, and $(1 + r)^{12} = 1.12$. Then $r = 1.12^{(1/12)} - 1 \; = \; .0094888$ or about 0.95% per month.

There is a general formula here. From the CIF, one has

$$(1 + r)^m \; = \; \frac{A_m}{A_0} \qquad \text{so} \quad m \; \ln(1 + r) \; = \; \ln A_m \; - \; \ln A_0$$

$$\therefore \quad 1 \; + \; r \; = \; \exp\left(m^{-1} \; [\ln A_m \; - \; \ln A_0]\right).$$

Sometimes this formula is approximated using the Taylor approximation for the exponential function which we will also use later. For x near zero one has

$$e^x \; = \; 1 \; + \; x \; + \; \frac{x^2}{2!} \; + \; \frac{x^3}{3!} \; + \; \frac{x^4}{4!} \; + \; \ldots.$$

So evaluate $X \; := \; m^{-1} \; [\ln A_m \; - \; \ln A_0]$ then the interest rate is approximately

$$R \; = \; X \; + \; \frac{X^2}{2} \; + \; \frac{X^3}{6} \; + \; \frac{X^4}{24}$$

Account Equations

Most accounts involve the payment of interest each period together with either deposits or withdrawals for an account during the period. For such accounts, the balance in each time period is

$$A_m \;=\; f_m\, A_{m-1} + c_m \qquad \text{for } m = 1, 2, 3, \ldots .$$

Here $f_m = 1 + r_m$ is the m-th growth factor and c_m is the contribution in the m-th time interval. When $c_m < 0$, then there is a withdrawal from the account.

In the following please **find, and write up for your own benefit, any algebra and calculus term or theorem that is used.** They will be needed repeatedly

For mathematical analysis we will just treat the case wher the interest rate and the contributions are constant. This holds if it is a savings account based on payroll deduction or a retirement account where there are equal contributions every time period.

Then the equation becomes

$$A_m \ = \ (1+r)\ A_{m-1} + c \qquad \text{for } m \geq 1$$

By repeated substitiution one sees that the solution of theis equation is

$$A_m \ = \ (1+r)^m\, A_0 \ + \ c\,[1+f+f^2+\ldots+f^{m-1}]$$

(Verify this for m=1,2,3 please.)

This may be simplified by using the formula for geometrical sums - which you should have seen in high school. it says that

$$1 + f + f^2 + \ldots + f^{m-1} \;=\; \frac{f^m - 1}{f - 1} \qquad \text{when} \quad f \neq 1$$

Question Find this sum when f=1 and then obtain the formula for A_m when r=0. Does your answer make sense?

For $r \neq 0$, the solution of a **uniform account equation** is

$$A_m \;=\; (1+r)^m A_0 \;+\; \frac{c}{r}\,[(1+r)^m - 1]] \quad \text{for } m \geq 1. \qquad (3)$$

For any value of r, the expression $(1+r)^m$ is a **polynomial** of degree m in r, that has the form

$$(1+r)^m \;=\; 1 + m\,r + \frac{m(m-1)}{2}\,r^2 + \frac{m(m-1)(m-2)}{6}\,r^3 + \ldots$$

from the **binomial theorem**. The ... indicates powers of r^4, r^5 and higher order powers when m is larger.

The **linear approximation** of a solution is

$$A_m \;=\; (1+r)^m\,A_0 + c\,m\left[1 + \frac{(m-1)}{2}\,r\right] \qquad (4)$$

The **quadratic approximation** of a solution includes the r^2 term and is

$$A_m = (1+r)^m A_0 + c\,m\left[1 + \frac{(m-1)}{2}\,r + \frac{(m-1)}{6}(m-2)\,r^2\right] \tag{5}$$

Example Suppose a person has an account that pays 1% per month. She makes and initial deposit of \$1,000 and then adds \$50 each month. What is the balance in the account after one year?

Comments: Any time you have a problem like this I suggest that you start by guessing an approximate answer. In this case, if no extra contributions were made, you know from the compound interest formula, that the \$1,000 will become more than \$1,126. Then she added \$600 in contributions. So the answer is probably larger than \$1,730 - thanks to the interest on her contributions. This is just a guess - and you may want to guess another number.

This account equation is to find A_{12} where

$$A_m \;=\; (1.01)\; A_{m-1} + 50 \;\text{ and }\; A_0 = 1000.$$

This has solution

$$A_m \;=\; (1.01)^m(1000) + \frac{50}{.01}[(1.01)^{12} - 1]$$

When m=12, this becomes, using the value of the powers here

$$A_{12} \;=\; 1126.825 + 5000(.126825) = 1126.825 + 634.125$$

or \$1760.95.

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or \$1760.95. Note that if the approximation had been used with just one power of r, there contributions would have yielded $50(12 + 66(.01)) = 633$. With 2 powers of r, the approximation becomes $50(12 + 66(.01) + 220(.0001) = 50(12.6820) = 634.1$ which is correct to less than 3 cents

Note that our guess and all the approximations are below the exact answer. Also notice that these formulae require the use of the binomial theorem from algebra and Taylor's theorem for approximating functions in calculus 1. These theorems will be used repeatedly in this class - and in financial calculations - so make sure that you know them.

The solution of an account equation gives A_m as a function of the 4 quantities, A_0, c, m, r. Thus it is an equation in 5 variables. If you know 4 of these quantities then you chould be able to find the fifth. I will give some homework problems of this type.

Another example is a simple college savings plan or other gift plan. Suppose your parents, grandparents or favorite aunt decides to save on a regular savings plan so they can give you a big gift on your 21st birthday, for your college expenses or whatever.

They invest in a program, or possibly buy an insurance policy, that will pay you \$10,000 at a certain time in the future provided you pay a certain amount $\$A_0$ now and contribute \$c per month. These savings accumulate at a certain interest rate, say 0.5% a month payable monthly.

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Usually the questions are either

(i) How much do they need to pay iniitialy if they can afford to contribute \$100 a month and it will be 60 months in the future? or

(ii) If they can put \$5,000 now into such an account what monthly payments do they have to make?

Answer (i) The problem is to find $\$A_0$ given that

$$m = 60, \quad A_{60} \;=\; 10,000, \quad c \;=\; 100 \text{ and } \quad r \;=\; 0.005.$$

The solution of the account equation (3) becomes

$$10000 \;=\; (1.005)^{60}\,(A_0 \;+\; 20000) \;-\; 20000$$

as $c/r = 20000$. Thus

$$A_0 \;+\; 20000 \;=\; 30000/1.34885 \;=\; 22241.17$$

or $A_0 \;=\; 2241.17$ is the exact answer for this problem. It may be rounded to \$2250 initially plus \$6,000 in total contributions + interest payments of about 6% per year for 5 years.

The linear approximation here leads to the equation

$$10000 \;=\; (1.005)^{60}\, A_0 \;+\; 6000\,(1 + 29.5(0.005))$$

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$$1.34885 A_0 \;=\; 10000 \;-\; 6885 \;=\; 3115$$

This yieds that $A_0 \;=\; 2309.37$ - which is a larger amount than the exact answer as we have undercounted the interest earned.

Similarly you can determine the value of A_0 from the quadratic approximation and this will be less than 2309 and more than 2242.

Examples such as these show that even the simplest interest theory problems are completely changed by the advent of computers and programs that guarantee the correct solutions using good mathematics for solving equations.

Moreover if you decide to change the interest rate or the number of years in these solutions, and you have a spread sheet - it is quite straightforward. The second homework will ask you to develop a spreadsheet that does these caculations for a problem like this. The important thing is to have the spread sheet - then you can plug in whatever are the relevant numbers and see what the answer is.

Answer (ii) The problem is to find $\$C$ given that

$$m = 60, \quad A_0 = 5000, \quad A_{60} = 10000, \text{ and } \quad r = 0.005.$$

The solution of the account equation (3) becomes

$$10000 = (1.005)^{60}\, 5000 + 200C(0.34885)$$

as $1/r = 200$. Thus

$$69.77C = 3255.75 \qquad \text{or} \qquad C = 46.66$$

To check this note that 60 payments of \$47 is \$2820 and together with the initial \$5000 they contribute \$7820 towards the 10K value at the end. This earns much more interest as it starts at 5K instead of less than 2.3K in the previous calculation.

Similarly for this problem one can ask about finding r or m when the other 4 variables are known. It is high school math to show how the equations can be "rearranged" so that there is the unknown on one side of the equation and numbers on the other side, so you just have to do a direct calculation.

Loan Amortization

The same formula is what is used to study how youy repay a loan. When you take out a loan, the balance on the loan satisfies an account equation with A_0 large and each payment corresponding to a negative value of c. If a borrower makes a payment of $P each time period and the interest rate is r, then the balance on a loan is given by

$$A_m = (1+r)\,A_{m-1} - P \quad \text{for } m \geq 1$$

The solution of this equation is given as above and usually one wants to know either when the loan is paid off of what the payment should be to pay off the loan in a given time - such as 2 years, 5 years etc.

The loan repayment formula says that if you have a loan of $\$L$ at an interest rate of r per time period, and make regular payments of $P, then after m time period the outstanding balance on the loan is, from the solution of the account equation

$$A_m = (1+r)^m L - \frac{P}{r}\left[(1+r)^m - 1\right] \quad \text{for } m \geq 1. \qquad (6)$$

Usually the questions about this loan is what should P be to pay the loan off in M payments, or what is the value of M such that this loan is paid off with regular payments of $P.

Note that this formula doesn't involve the actual time period. The answers are the same whether we take the time period to be days, weeks, months or years!

If the loan is paid off in M time periods, then $A_M = 0$ so one has

$$\frac{P}{r} \left[(1+r)^M - 1 \right] \;=\; (1+r)^M \, L \tag{7}$$

That is

$$P \;=\; p(r,M) L \quad \text{with } p(r,M) \;=\; \frac{r\,(1+r)^M}{[(1+r)^M - 1]} \tag{8}$$

This says that the payment is linear in the loan amount L - which makes sense but it is a complicated function of r and M. This function is an increasing function of r for fixed M and a decreasing function of M for fixed r.

This is proved by evaluating the partial derivative of $p(r, M)$ with respect to r,M respectively. Suggest that you try this, we'll look at this later.

The next few slides are to remind you of some calculus results that will be used a lot. Suggest that you review the reasons why they hold and perhaps try to see which apply to various formulae described so far.

Monotone and convex functions

Suppose that a function $f(x)$ is defined for $a \leq x \leq b$ and let $I = [a, b]$ be this interval. The function f is said to be

(i) **increasing** on I provided $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

(ii) **strictly increasing** on I provided $f(x_1) < f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

(iii) **decreasing** on I provided $f(x_1) \geq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

(iv) **strictly decreasing** on I provided $f(x_1) > f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.

In calculus I you should have learnt that when f is differentiable on the open interval (a,b), then

(a) the function f is increasing (decreasing) on I whenever $f'(x) \geq (\leq) 0$ for $x \in (a, b)$.

(b) the function f is strictly increasing (strictly decreasing) on I whenever $f'(x) > (<) 0$ for $x \in (a, b)$.

The function is (strictly) monotone on I if it is either (strictly) increasing or decreasing on I.

The function f is said to be **convex** on I provided for all $x_1, x_2 \in I, \quad 0 \leq t \leq 1,$

$$f((1-t)\,x_1 + t\,x_2) \;\leq\; (1-t)\,f(x_1) + t\,f(x_2)$$

This says that the function $f(x)$ lies below the straightline joining the points on the graph of f at x_1, x_2.

Evaluate the derivatives to show that a quadratic function $f(x) \; := \; ax^2 + bx + c$ is convex if $a \;\geq\; 0$, increasing if it is convex and $b \geq 0$. What are the conditions that f be strictly increasing?

The function $f(x) \; := \; a^x$ is strictly increasing and convex when $a > 1$.

In general when f is continuously differentiable and the derivative $f'(x)$ is increasing on (a, b), then f is convex.

When f is twice continuously differentiable and $f"(x) \geq 0$ on $a < x < b$, then f is convex on I.

The function f is **concave** on I when each $\leq$ is replaced by $\geq$ above. Alternatively f is concave when $-f$ is convex.

The function $l(x) := \log_a x$ is strictly increasing and concave when $a > 1, x > 0$. For many of the formulae in finance and the theory, it is important to prove inequalities for the values of functions and their derivatives - as you will see. You just need to know how to calculate the derivatives.

Example. Show that the doubling time for a simple savings account with interest rate r is strictly decreasing and convex.
Proof. The doubling time is given by the function $g(r) := \ln 2 / \ln(1+r)$ with $r > 0$. Evaluate the derivative

$$g'(r) = \frac{dg}{dr}(r) = \dots$$

Observe that this derivative is always negative and you can verify that $g'(r)$ is an increasing function of r. Since the function $\ln(1+r)$ is strictly increasing, g is strictly decreasing.

If you are an ace at calculus, you might get the right answer for the second derivative of $g(r)$. Suggest that you use a spreadsheet or a program to graph this function - and its first derivative to show that the function is convex.

Very often with compound interest the interest rate changes each time period, and is given by

$$A_m = (1 + r_m)\, A_{m-1} = f_m\, A_{m-1} \quad \text{for } m \geq 1.$$

Then the first few values are

$$A_1 = f_1 A_0, \; A_2 = f_2\, f_1\, A_0, \; A_3 = f_3\, f_2\, f_1\, A_0, \quad \ldots$$

In general

$$A_m = f_m\, f_{m-1} \ldots, f_2\, f_1\; A_0$$

Often we replace all these different growth factors by a single average growth factor to do calculations. This growth factor will be the solution of

$$f^m = f_m \, f_{m-1} \, \dots, f_2 \, f_1$$

This number is called the geometric mean of the positive values $f_1, \dots, f_m$ and is given by

$$f_g := [f_1 \, f_2 \, \dots, f_{m-1} \, f_m \,]^{1/m}$$

It is easy to prove that for 2 positive numbers x_1, x_2

$$\sqrt{x_1 \, x_2} \leq (x_1 + x_2)/2$$

or the geometric mean of two numbers is less that the usual mean value.

This geometric mean is an equivalent growth factor for which we can use the formulae for constant (or uniform) growth. When each $f_m = 1 + r_m$, the equivalent uniform interest rate is r_g where

$$f_g = 1 + r_g$$

The solutions of the uniform interest payments case with the equivalent uniform rate and the variable interest rate cases usually are very close. So when there are variable interest rates, people often just evaluate the equivalent uiform rate and use the formulae we've obtained earlier.

How good this is, requires a lot more calculus which is why mathematicians are useful for some calculations in finance.

The **arithmetic-geometric mean inequality (AGM)** is the fact that for any M positive numbers $a_1, \ldots, a_M$, this still holds. That is, the geometric mean of M positive numbers is less than the arithmetic mean unless they all are equal.

$$a_g \ := \ [a_1\, a_2 \ \ldots, a_{M-1}\, a_M \ ]^{1/M} \ \leq \ \frac{1}{M}[a_1 + a_2 \ \ldots + a_{M-1} + a_M]$$

The arithmetic mean will usually be denoted

$$\overline{a} \ := \ \frac{1}{M} \ [a_1 + a_2 \ \ldots + a_{M-1} + a_M]$$

so the AGM inequality is that $a_g \ \leq \ \overline{a}$. Sometimes one also use the obvious fact that

$$a_g \ \geq \ \tilde{a} \qquad \text{with } \tilde{a} \ = \ \min\{a_1, \ldots, a_M\}.$$

when you want to check a calculation.

Example. Suppose that a dealer lends you \$10,000 to help pay for a car. You have good credit, so he charges a nominal interest rate $0.06 = 6\%$ per year payable monthly for 2 years. How much will each payment be? What is the total payment and how much interest will you pay?

First note that the paperwork should tell you this loan has APR $0.0617 = 6.17\%$. Why this rate? Use the formula with $M = 24$, $r = 0.005$ then $(1.005)^{24} = 1.12716$ so

$$p(r, M) = \frac{0.0056358}{0.12716} = 0.0443205$$

This each payment is $P = 443.21$

Your total payment is $24P = 10636.93$ and the total interest payment is \$636.93.

Earlier we used the fact that

$$(1+r)^M \;=\; 1 \;+\; Mr\; \left[1 \;+\; br \;+\; cr^2 \;+\; \ldots\right]$$

with $b \;=\; (M-1)/2,\;\; c \;=\; (M-1)(M-2)/6$ and said that usually in calculations we only need these terms in most evaluations. The formula for the M uniform payments P on a loan of L dollars with interest rate r per time period becomes, when br, cr^2 can be neglected

$$P \;=\; p_0(r,M)\; L \qquad \text{with } p_0(r,M) \;=\; r + \frac{1}{M}$$

In the previous example this is 0.0466.67 for payments of 466.67 per month.

The linear approximation, which neglects cr^2 is

$$P = p_1(r, M)\ L \qquad \text{with } p_1(r, M) = r + \frac{1}{M(1 + br)}$$

The quadratic approximation is

$$P = p_2(r, M)\ L \qquad \text{with } p_2(r, M) = r + \frac{1}{M(1 + br + cr^2)}$$

Since $b, c > 0$ when $M \geq 3$ these factors decrease as more terms are included.

You can do the calculations for the auto loan above. For each of these approximations, you may check to see if they are increasing functions of r and decreasing fuctions of M

In view of these formulae, you can also look at questions such as, if you can only afford to make payments of about $360 a month on the car, then how many payments will you have to make?

You can write an algorithm to solve this exactly. However could also. just take the simplest approximation and see that at this interest rate you want

$$(.005 + 1/M)\ 10000 \leq 360$$

This yields $M = 62.5$ so it seems that a 5-year loan with 60 payments should work. Suggest that you evaluate the exact cost of such a loan.

Similarly, if the dealer says that they will lend you the funds with repayments of \$450 per month for 24 months, then you know that they are charging you a slightly higher APR than $0.0617 = 6.17\%$.

If you want to find the interest rate that is being charged then you would have to solve the equation

$$\frac{r\,(1+r)^M}{(1+r)^M - 1} \;=\; \frac{P}{L} \;=:\rho$$

Here ρ is the proportion of the regular payment to the original loan amount. Note that you must have $MP > L$ when the interest rates are positive, so $\rho > 1/M$ and the interest rate is a positive solution of the equation

$$(1+r)^M(r \;-\; \rho) \;+\; \rho \;=\; 0$$

Let $x := 1 + r$, then this equation becomes

$$f(x) := x^{M+1} - (1+\rho)\, x^M + \rho = 0 \qquad (9)$$

This is the **interest rate equation** for a loan. To find the interest rate, solve this equation for a solution $\hat{x}$ near 1 and then the interest rate will be $\hat{r} := \hat{x} - 1$.

There are algebraic formulae for the solutions of these equations when M=1,2 and 3 but not in general for $M \geq 4$. However, $\hat{x} = 1$ is a solution with $f'(1) < 0$ when $\rho > 1/M$. Thus $f(x)$ is below 0 for $x > 1$ and close to 1. It is easy to show that there is no solution of this equation larger than $1 + \rho$. Some more calculus shows that there is a unique solution with

$$x_M := \frac{M(1 + \rho)}{M+1} < \hat{x} < 1 + \rho.$$

as $f(x)$ is strictly increasing for $x > x_M$.

Computing Zeros of Equations.

The simplest algorithm for solving the equation $f(x) = 0$, when one knows values x_1, x_2 with $f(x_1) < 0 < f(x_2)$ is the **bisection method.** It is guaranteed to yield a solution when f is continuous on $I := [x_1, x_2]$ and it will be the unique solution if $f'(x) > 0$ on I. This is the intermediate value theorem from calculus I.

Usually one just seeks to find a value $\tilde{x}$ with $|f(\tilde{x})| < \epsilon$where $\epsilon > 0$ is a small number. If one want $f(x)$ to be zero to 3 decimal places, take $\epsilon = 0.0005$. Sometimes we just want the answer to some accuracy δ.

The Bisection Algorithm.

The bisection algorithm is given $I_1; = [x_1, x_2], f$ and δ, ϵ as above,
Step 1. For $j \geq 1$, $I_j := [x_1, x_2]$, evaluate $|I_j| = x_2 - x_1$, $\xi = (x_1 + x_2)/2$, and $f(\xi)$.
Step 2. If $|I_j| < \delta$, or $|f(\xi)| < \epsilon$ take ξ to be the solution and **stop**.
Step 3. Otherwise when $f(\xi) \geq \epsilon$, replace x_2 by ξ so that $I_{j+1} = [x_1, \xi]$. If $f(\xi) \leq -\epsilon$, take $I_{j+1} = [\xi, x_2]$ and **go to step 1**.

With this algorithm, one sees that $|I_{j+1}| = |I_j|/2$ so the interval containing a zero decreases in size by a factor of 2 at each step. It guarantees that in a finite number of steps either $|I_j| < \delta$, or $|f(\xi)| < \epsilon$, so we have found an approximate zero of the function $f(x)$.

It is quite easy to implement a spreadsheet program to use the bisection method for solving an equations such as the interest rate equation. Note you only have to evaluate the function $f(x)$ - no derivatives or any other functions are required. Please write you own spread sheet program to solve equations of the form $(1+x)^m \ = \ 2$ with m=2 -10. These provide formulae for doubling times for a simple compound interest problem. The first step is to make good choices for x_1, x_2. Choose sime good guesses.

if you have that when $x_1 < x_2$ one has $\quad f(x_1) > 0 > f(x_2)$ so there still is a zero in the interval, then use $g(x) := -f(x)$ and the function g will have the properties required for the bisection algorithm to work.

Present Value of a future Payment

People value a payment of $P more highly today than the promise of a payment of the. same amount at some time in the future. Also it is worth more if it will be1 week in the future compared to 1 month in the future, 1 year in the future or 5 years in the future. That is the longer you have to wait the lower the **present value PV**.

Suppose it costs $ PV today to buy a contract for an amount of $ A in T years time, then the **discount rate d(T)** of this contract is

$$d(T) := \frac{PV}{A} \qquad \text{or} \qquad PV = d(T)\,A$$

Here PV is the **present value** of the payment $ A in T years time and d(T) is usually between 0 and 1.

The **annual discount rate** of this contract is

$$d_a := d(T)^{1/T}$$

Discount rates measure the **time value of money** and may be related to interest rates. An annual discount rate corresponds to an APR of r_a where

$$d_a = \frac{1}{1+r_a} \quad \text{or} \quad r_a = \frac{1-d_a}{d_a}$$

Discount rates may also are quoted in terms of weeks, months or other time periods.

See Wikipedia, or other internet sites, for descriptons of these ideas with many different notations and conventions. The important aspect is that in these contracts, the amount to be returned at time T is the fixed amount \$A and the "price" is the immediate cost of this promise - that is, the present value.

Example. Four grandparents want to make a gift of \$10,000 cash to a grandchild on her 21st birthday in 3 years time. Some are in ill-health and are not sure that they will be around, or have the money then, so they pay \$9,700 for a certificate of deposit in her name that will be worth \$10,000 when it matures 3 years later.

Here the 3-year discount rate is $d(3) = 0.97$ so the annual discount rate is 0.98990 which corresponds to earning an APR of $0.0111 = 1.11\%$. Suggest that you check this, and all similar evaluations so you know what formulae are used.

Example. A company has just made a big sale and realizes that it will have to pay the governent at least a million dollars in taxes in 6 months time. The cost of a 6-month T-bill is \$990 so it buys 1000 such bills to ensure that it can pay this amount. What is the annual discount rate, and APR, of these bills?

The discount rate of these bills is $d(1/2) = 0.99$, the annual discount rate is $d_a = 0.99^2 = 0.9801$ and r_a is 2.03%..

These contracts are also called a single premium annuity and may be arranged with banks, various brokers and some government agencies as well as with individuals and companies. The federal government sells **T-bills** in multiples of a thousand dollars and for different time lengths via auctions.

If you search Wikipedia, or the internet for annuities, certificates of deposit, zero-coupon bonds or similar keywords you will find many different businesses that will be pleased to take your money now in exchange for some promise to pay you back in the future.

Your primary concern should be whether they will actually do so - or whether they will “default”. Government issued contracts (T-bills, bonds, notes) are regarded as “riskless” or “safe” - since the government also ”prints the money” or ”controls the currency”. However even governments occasionally default on their promises; particularly after wars.

Annuities

An annuity is a series of payments at specified times in the future. Examples include

the payout of lottery prizes as amounts of \$P each year for 20 or 30 years,

Divorce, injury or other legal payments of \$P a month for a specified number of months.

Many people buy retirement annuities that pay $\$P$ each month, or quarter or ... for a fixed number of years starting at age 70 or some other age.

Many annuities are " life annuities" that make payments for the rest of a person's life (such as social security payments and many retirement programs). The mathematics of those depend on the probability that a person will lives to various ages - so they should use life tables and need probability theory for the calculations. This is a big business for insurance companies.

If you search for annuity on the internet there are lots of provider advertising and also of people who will advise you (for a fee) on possible annuities to buy.

There also are many **annuity calculators** which essentially just are small programs that do the calculations that I'll describe here. A good example is the *Present Value of Annuity Calculator* at *financialmentor.com* who say they provide "Financial freedom for Smart Prople".

The following slides will describe the models and formulae behind the simpler examples. Often people want to add extra conditions and requirements to a "boilerplate" policy.

Suppose an annuity consists of M payments of \$ A starting now and paid N times a year. An insurance company prices the annuity at a uniform discount rate of d_a per year. Then the m-th payment is due at time $t_m := m/N$ years from now and has present value

$$PV_m \;=\; (d_a)^{m/N}\, A \;=\; d^m\, A$$

where $d = (d_a)^{1/N}$ is the discount rate per payment.

Then the present value of the annuity is the sum of each individual annuity so if it starts after $1/N$ years, then

$$PV \;=\; PV_1 + PV_2 + \ldots + PV_M$$

$$PV \;=\; \left[d + d^2 + \ldots + d^M\right]\, A \;=\; \frac{d\,(1 \,-\, d^M)}{1-d}\, A$$

using the formula for geometric sums.

More generally if the payment starts after K time periods from now, then

$$PV = \sum_{m=0}^{M-1} d^{K+m} A$$

and this sum is

$$PV = \frac{d^K (1 - d^M)}{1 - d} A$$

Here K could be any positive number or even zero and measures the "waiting time" to the first payment. These are the **Annuity Equations** that say how much an annuity should cost at a given discount rate.

The annuity equation may be written

$$PV = v_{MK}(d)A \quad \text{with } v_{MK}(d) := \frac{d^K (1 - d^M)}{1-d}$$

Here $v_{KM}(d)$ is called the **annuity factor**. It is a decreasing function of K, M and an increasing function of d for $0 < d < 1$. When M is very large, the distant (m large) payments have very small present value. If $M = \infty$ these are called **perpetuities**, that have an infinte payout but finite present value!

The present value of a perpetuity of $A per payment, starting K time periods from now at a discount rate d is

$$PVP := \frac{d^K}{1-d} A$$

These are promises to pay “forever” and are not legal in most states and countries but are often used for pricing annuities with M very very large such as for 100-year leases for property.

Note that a lease with uniform fixed payments at fixed times (weekly, monthly, quarterly, etc) is a simple annuity. Usually to evaluate the present value of a lease, you need to treat deposits separately.

Example. What is the cost of an annuity that pays $1000 every four weeks for 52 weeks starting in a year's time if it is bought at an annual discount rate of 0.96? The cost uses the preceding formulae evaluated with 4 week periods. The discount rate each four weeks is $d = 0.96^{1/13} = 0.996865$. Then

$$PV = 960 \frac{0.04}{.003135}. = 960(12.75917) = 12248.80.$$

That is, an insurance company offering this annuity should quote a price of $12,248.80 to the buyer. They will then pay out a total of $13,000 to the recipient of the annuity

Often this is the way "trust fund babies" are paid or patients in a nursing home. The trust fund buys such an annuity, then the insurance company makes this arranged payout to the designated beneficiary.

Continuous Interest rates

We have seen that a nominal interest rate of r per year charged daily leads to a daily interest rate of $(r/365)$ so the APR is given by

$$1 + r_a = \left(1 + \frac{r}{365}\right)^{365}$$

since the daily growth factor is $f_d = (1 + \frac{r}{365})$.

Example A nominal interest rate of $r = 0.1825$ compounded daily has an APR of $r_a = .20016$ or 20.016%.

One of the definitions of the exponential function is that

$$e^x := \lim_{m \to \infty} (1 + x/m)^m$$

Thus a nominal annual interest rate of r compounded continuously (that is every second of every day) will yield an annual percentage rate of $r_a := e^r$ or a growth factor of $r(T) = e^{rT}$ over T years.

Example. The nominal interest rate of 18.25% per year compounded continuously has an APR of $r_a = .20021$ which is only very slightly higher than daily compounding.

We say that an account earns an **annual continuous interest rate** r_c provided its growth factor after time T is

$$f(T) \; := \; e^{r_c T} \qquad \text{or} \quad A(T) \; = \; A(0)\, e^{r_c T}$$

Thus an account that earns interest with an APR of r_a per year has **continuous interest rate** $r_c \; := \; \ln(1 + r_a)$ as the annual growth rates are $f \; = \; 1 \; + \; r_a \; = \; e^{r_c}$

This says that an annual rate of r_a paid once a year is equivalent to an interest rate of r_c every second during the year (to many decimal places) as they have the same 1-year growth factors.

Example. What is the continuous interest rate corresponding to an APR of $0.1 = 10\%$?

Ans: $r_c \; = \; \ln 1.1 \; = \; 0.095310$

When interest is compounded more often than weekly, most people use continuous compounding since it simplifies many of the mathematical formulae.

The continuous rate always obeys $r_c < r_a$ as $\ln(1+x) < x$ for $x > 0$. From the Taylor series for the exponential function one sees that

$$r_a \;=\; r_c \;+\; \frac{1}{2}\,{r_c}^2 \;+\; \frac{1}{6}\,{r_c}^3 + \ldots$$

so r_c will be close to (and below) the solution of

$$x^2 \;+\; 2\,x \;-\; 2\,r_a \;=\; 0.$$

This is the equation you get from just using 2 terms in the Taylor series. A better approximation is the solution that uses 3 terms and is a cubic equation for an approximation of r_c.

Just as there is an APR associated with any discrete interest rate so also there is a continuous interest rate associated with any discrete interest rate. Suppose r_2 is an interest rate per $1/2$ year, r_4 is an interest rate per quarter, r_{12} is an interest rate per ordinary month and r_{52} is a weekly interest rate. Then the continuous interest rate r_c associated with these rates satisfies

$$e^{r_c} = (1+r_2)^2 = (1+r_4)^4 = (1+r_{12})^{12} = (1+r_{52})^{52}$$

Each of these is the 1-year growth factor at the indicated rate.

Exercise What is the similar formulae for the 4-week interest rate r_{13} and the daily interest rate r_{365}? Complete the following formulae for r_{13}, r_{52}, r_{365}.

$$r_c = 2 \ln(1+r_2) = 4 \ln(1+r_4) = 12 \ln(1+r_{12}) = \ldots$$

When a continuous interest rate r is known, then the growth factor of an investment of $A for time T is

$$f(T) \; := \; e^{rT} \qquad \text{so} \quad A(T) \; = \; A \, e^{rT}$$

The present value of a payment of $A to be made at time T in the future is

$$PV \; = \; d(T)\, A \quad \text{with} \quad d(T) \; = \; e^{-rT} \; = \; \frac{1}{f(T)}$$

US Treasury Bills or **T-bills**

See Wikipedia entry for United States Treasury security.

These are issued by the US government who promise to pay you $1,000 per bill in 4, 8, 13, 26 or 52 weeks time from the date of purchase. An M week bond has a **term** or **time to maturity** of M weeks.

Buyers pays $10 P per bill with $P < 100$ being determined by an auction. Since these are government bonds they are considered to be "riskless" and have the lowest interest rates of any bonds available in the US for this time period. So T-bill rates are used for comparison purposes to all other interest rates.

Suppose that an M week T -bills is sold for a price of \$10P, then we say that the M-week treasury (continuous) interest rate is r_k per year where

$$P = 100e^{-(k\, r_k)} \text{ and } \quad k = 52/M.$$

Thus the 52-week T-bill rate is the annual continuous interest rate associated with this purchase. The 26-week T-bill rate is the semi-annual continuous interest rate per year The 13-week T-bill rate is the quarterly continuous interest rate and the 4-week T-bill rate is the (lunar) monthly interest rate per year.

A plot of interest rate against time to maturity, or term, is called a **yield curve**.

Example 1. If a 12-month T-bill costs \$980, then we say its price is P = \$98.00. The 52-week T-bill continuous rate is r_a where

$$e^{r_a} = 100/98 = 1.0204082. \quad \text{Thus}$$

$$r_a = \ln 1.0204082 = 0.02020271$$

This is a continuous rate of 2.02% per year.

Example 2. A 6-month T-bill costs \$991.00 so its price is said to be P= \$99.10. The 26-week T-bill continuous rate is r_2 where

$$e^{r_2} = 100/99.1 = 1.0090817 \quad \text{Thus}$$

$$r_2 = 2 \ln 1.0090817 = 0.0180815$$

This is a continuous rate of 1.808% per year.

Zero Coupon Bonds

T- bills are special examples of what are called zero coupon bonds (ZCB). These are bonds that do not make any interest payments but which will pay a given amount \$A at time T from now. A is called the **face value** of this bond and T is the term or the time to maturity. The present value of a bond with face value \$A, at the continuous interest rate r is

$$PV = A\, e^{-rT}$$

We say the price of such a bond is $P = 100\, e^{-rT} < 100$ and assume that $r > 0,\ T > 0$. So the price is the cost of such a bond with face value \$100 at maturity.

Example. What is the price of a ZCB that pays a continuous interest rate of 3% and has 3 years to maturity?
Answer. $P = 100\ e^{-0.09} = \$91.3931$. Thus each \$1,000 bond will cost \$913.93.

Problem A 4-year ZCB is bought for \$900. What is the continuous interest rate for this bond?

This time want to find r, when $900 = 1000\ e^{-4r}$. .
Thus $e^{4r} = 1.111111111$ so $4r = .1053605$ and $r = 0.02634$ or the interest rate is 2.634%.

In this problem, the solution for r is often called the **internal rate of return** (IRR) or the **yield** on this bond. This IRR is then denoted by y (for yield).

Usually one has that the interest rate on shorter term bond from a company or government to be less than that of a longer term bond so one expects the yield curve to be an increasing function of the time-to maturity (or term). If there is a range of "terms" where the yield decreases as the term increases then one has an " inverted" yield curve and a trader could make a profit using riskless "arbitrage" . This could happens when there is likely to be a big change between the maturity dates such as a declaration of war or a difficult election.

Banks usually pay interest rates on savings and CDs that are below the annual interest rates on T-bills - as they often purchase T-bills with the funds. On Wikipedia, and in older texts they describe the "discount yield" of a T-bill. This is a number that is very, very close to this continuous yield per year.