Additions and corrections for
Variational Principles for Finite Dimensional
Initial Value Problems.

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Abstract. The following notes provide some corrections and material to complete the proof
of theorem 6.2. of the paper which appeared in Contemporary Mathematics, 426, pp 45-56.

Equation (2.1) should read.

\[ 0 \in \dot{u}(t) + \partial_u V(t, u(t)) \quad \text{for all } t \in [0, T]. \]

The material around Equation (6.6) should read

\[ L : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \text{ is defined by} \]

\[ L(t, u, z) := q^*(t, z + B(t)u - f(t)) + q(t, u) - \langle f(t), u \rangle. \]

The second paragraph of theorem 6.2 was a stub whose content should be replaced
by the following result which proves the coercivity of \( J \) on a closed convex subset of
\( H^1((0, T); \mathbb{R}^m). \)

Lemma 6.3. Assume (A1) holds and \( J \) is defined by (6.5)-(6.6), then \( J \) is coercive on
\( K_0. \)

Proof. Write the Lagrangian (6.7) as

\[ 2 \mathcal{L}(t, u, z) = l_2(t, u, z) + l_2(t, u, z) + l_0(t) \]

where \( l_2 \) is the quadratic part

\[ l_2(t, u, z) := \langle A_S(t)u, u \rangle + \langle A_S(t)^{-1}(Bu + z), Bu + z \rangle \]

of \( \mathcal{L} \), while \( l_1 \) includes the terms that are linear in \( u \) or \( z \) and

\[ l_0(t) := \langle A_S(t)^{-1}f(t), f(t) \rangle. \]

Introduce the family of inner products

\[ [x, y]_t := \langle A_S(t)^{-1}x, y \rangle \quad \text{for } t \in [0, T]. \]
For each $t$, (6.4) shows this is an equivalent inner product to the usual inner product on $\mathbb{R}^m$. The standard inequality

$$2[z, Bu]_t \leq \epsilon \|z\|_t^2 + \epsilon^{-1}\|Bu\|_t^2 \quad \text{for all } z, u \in \mathbb{R}^m$$

holds for any $\epsilon > 0$. Thus the expression $l_2$ obeys

$$2 l_2(t, u, z) \geq a_0\|u\|^2 + c\|z\|^2 - c\epsilon^{-1}\|Bu\|^2_t \quad (0.0.1)$$

where $c = 1 - \epsilon$. From (A0) and (A1), since the matrices are uniformly bounded on $[0,T]$, there is a constant $b > 0$ such that

$$\|Bx\|_t \leq b \|x\|_2 \quad \text{for all } t \in [0, T], x \in \mathbb{R}^m.$$

Choose $\epsilon$ so that

$$\frac{b^2}{a_0 + b^2} < \epsilon < 1.$$

Substitute in (0.0.1), then

$$2 l_2(t, u(t), \dot{u}(t)) \geq c \|\dot{u}(t)\|^2_t + c_1 \|u(t)\|_2^2 \quad (0.0.2)$$

where $c > 0$. The other terms in the Lagrangian (6.7) grow at most linearly in $u, \dot{u}$, so this implies that the functional $J$ defined by (6.5) is coercive on $K_0$.

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