L^2-WELL-POSEDNESS OF 3D *div-curl* BOUNDARY VALUE PROBLEMS

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Abstract. Criteria for the existence and uniqueness of weak solutions of *div-curl* boundary-value problems on bounded regions in space with C^2-boundaries are developed. The boundary conditions are either given normal component of the field or else given tangential components of the field.

Under natural integrability assumptions on the data, finite-energy (L^2) solutions exist if and only if certain compatibility conditions hold on the data. When compatibility holds, the dimension of the solution space of the boundary-value problem depends on the differential topology of the region. The problem is well-posed with a unique solution in $L^2(\Omega;\mathbb{R}^3)$ provided, in addition, certain line or surface integrals of the field are prescribed. Such extra integrals are described.

These results depend on certain weighted orthogonal decompositions of L^2 vector fields which generalize the Hodge-Weyl decomposition. They involve special scalar and vector potentials. The choices described here enable a decoupling of the equations and a weak interpretation of the boundary conditions. The existence of solutions for the equations for the potentials is obtained from variational principles. In each case, necessary conditions for solvability are described and then these conditions are shown to also be sufficient. Finally L^2-estimates of the solutions in terms of the data are obtained.

The equations and boundary conditions treated here arise in the analysis of Maxwell's equations and in fluid mechanical problems.

1. Introduction

This paper treats existence and well-posedness issues for boundary value problems for *div-curl* systems of the form (2.1)–(2.2) on bounded 3-d regions. In particular we treat situations where

(i) the normal component of the field is prescribed on the boundary, or
(ii) the tangential components of the field is prescribed on the boundary.

Each of these cases each arise in electromagnetic modelling and we show that

(i) the data must satisfy certain compatibility conditions for solutions to exist,
(ii) depending on the topology of the region and the boundary data, the solutions may be non-unique, and
(iii) the solutions are described by variational principles for specific scalar and vector potentials.

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The specific well-posedness results are Theorems 12.6 and 13.4 of this paper. These theorems are based on the representation theorem 5.2, and much of this paper is devoted to establishing this weighted *Hodge-Weyl* decomposition. A major issue is the description of appropriate classes of potentials for each of these problems. When the potentials are selected appropriately, the system may be decomposed into independent problems for the scalar potential and the vector potential of the solution.

Variational principles for these potentials are described and the existence and uniqueness of solutions is analyzed. For each of these problems, we first describe the necessary conditions on the data for the existence of solutions. Then we show that, subject to some further natural integrability requirements on the data, these necessary conditions are, in fact, sufficient to guarantee the existence of finite-energy solutions. When the topology of the domain or the boundary data is non-trivial, there may be non-uniqueness of solutions of the pure boundary value problem. This non-uniqueness is manifested in the existence of a finite dimensional subspace of ε-harmonic vector fields. Extra linear functionals must then be specified to determine a unique solution. Such extra conditions are detailed.

For the prescribed flux problem these extra conditions may often be interpreted as requiring the prescription of extra *circulations*. When the tangential component of the field is prescribed, the extra conditions may be regarded as *potential differences*. The number and type of such extra conditions are independent of the tensor $\varepsilon(x)$ in the equations, provided condition E1 holds. In the case of prescribed flux, the number of extra conditions is the number of handles in the domain - or the dimension of the first de Rham cohomology group. When the tangential field is prescribed on the boundary, the number of potential differences required is the number of holes in the domain - or the dimension of the second de Rham cohomology group of the region.

Earlier work on the normal and tangential boundary-value problems for these systems includes that by Picard [16] and Saranen [17] and [18]. Their results used the theory of closed linear operators on Hilbert spaces. Recently Bolik and von Wahl [8] have described a C^α approach to these questions. Cessenat, in chapter 9 of [9], has provided a careful analysis of these problems. The variational approach detailed here is quite different to any of the preceding works and provides, we believe, both a physically reasonable, and an numerically advantageous formulation of these problems. In particular, it permits the treatment of boundary conditions in certain natural weak forms. Previously, Kotiuga and Silvester in [15] described a variational principle related to those described here for a modelling a magnetostatic problem.

The results of this paper could have been formulated in the language of exterior analysis and differential forms. This was not done here for a variety of reasons including the fact that using the same exterior derivative symbol for each of grad, curl and div ends up being quite confusing. Moreover the analysis given here applies to contravariant fields such as the velocity field in inviscid fluid mechanics as well as to covariant fields. It is, however, very useful to interpret many of the results for the special case of forms on open sets in $\mathbb{R}^3$.

This paper has many similarities with our paper on the planar *div-curl* problem [6], although there are many changes in the notation, results and methodology because of the extra dimension. In both papers, however, the results are obtained by using potential decompositions of finite-energy fields and variational principles for the potentials. Also Auchmuty [4] described related principles for finding the velocity from the vorticity in 3-dimensional fluid flows.

2. *div-curl* Equations on Bounded Regions.

This section describes the boundary-value problems for *div-curl* systems that are be studied. Throughout this paper, Ω is a non-empty, bounded, connected open subset of $\mathbb{R}^3$. Its closure is denoted by $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. A non-empty, connected, open subset of $\mathbb{R}^3$ is called a *region*. We generally require:

Condition B1. *Ω is a bounded region in $\mathbb{R}^3$ and $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces; each surface having finite surface area.*

A closed surface Σ in space is said to be C^2 if it has a unique unit outward normal ν at each point and ν is continuously differential vector field on Σ. See [14], Section 1.1. for more details on this definition. When (B1) holds and $\partial\Omega$ consists of $J+1$ disjoint, closed surfaces, then J is the *second Betti number* of Ω, or the dimension of the second de Rham cohomology group of Ω. Geometrically it counts the number of "holes" in the region Ω.

When u, v are vectors in $\mathbb{R}^3$, their scalar product, Euclidean norm and vector product are denoted $u \cdot v$, $|u|$, and $u \wedge v$, respectively. The issue to be studied here is:

Given a Lebesgue-integrable real-valued function ρ, a vector field ω defined on Ω and specific boundary conditions, when does the system

$$\operatorname{div}\big(\varepsilon(x)v(x)\big) = \rho(x) \quad \textit{and} \tag{2.1}$$

$$\operatorname{curl} v(x) = \omega(x), \quad \textit{for } x \in \Omega, \tag{2.2}$$

have a unique weak solution $v \in L^2(\Omega;\mathbb{R}^3)$?

Here div and curl are the usual vector differential operators and $\varepsilon(x) := \big(e_{jk}(x)\big)$ is a symmetric, positive definite, 3×3 matrix for each $x \in \overline{\Omega}$. The matrix $\varepsilon(x)$ is required to satisfy the following condition:

Condition E1. *Each component e_{jk} of ε is continuous on $\overline{\Omega}$ and there exist positive constants e_0 and e_1 such that, for all $x \in \Omega$ and $u \in \mathbb{R}^3$,*

$$e_0\,|u|^2 \;\le\; (\varepsilon(x)u)\cdot u \;\le\; e_1\,|u|^2 . \tag{2.3}$$

The system (2.1)–(2.2) arises in fluid mechanics and electromagnetic field theory, as well as other applications. For a fluid , $\varepsilon(x)$ is a diagonal matrix whose diagonal entries are the local mass density. In electrostatics, $\varepsilon(x)$ is the permittivity matrix. In linear magnetic field theory, $\rho(x)$ is zero, v represents the magnetic field intensity and then

$\varepsilon(x)$ is the inverse of the magnetic permeability tensor. Much of the following analysis is motivated by questions about the solvability of Maxwell's equations. The conditions imposed, and the results obtained, have physical interpretations in electromagnetic field theory.

The normal (or prescribed flux) boundary-value problem for (2.1)–(2.2) is given a function μ on $\partial\Omega$, to find solutions v subject to

$$\big(\varepsilon(x)v(x)\big) \cdot \nu(x) \;=\; \mu(x) \text{ on } \partial\Omega. \tag{2.4}$$

The tangential boundary-value problem is given a tangential vector field η on $\partial\Omega$ to solve (2.1)–(2.2) subject to

$$v(x) \wedge \nu(x) \;=\; \eta(x) \text{ on } \partial\Omega. \tag{2.5}$$

The field η is *tangential* on $\partial\Omega$ provided

$$\eta(x) \cdot \nu(x) \equiv 0 \text{ on } \partial\Omega. \tag{2.6}$$

This paper develops sharp conditions under which the *div-curl* system (2.1)–(2.2) has finite-energy (L^2) weak solutions which satisfy the boundary conditions in a specific weak sense. An important part of this paper is to establish this weak formulation for this problem. The guiding principle in this work is the use variational principles to determine both the formulation, and the solutions, of these boundary value problems. To do this special potential representations of the solutions are introduced. These separate into distinct problems for the scalar and vector potentials of the solution. The next two sections describe the various Hilbert spaces used in these decompositions which allows us to state the *weighted Hodge-Weyl* representation in Theorem 5.2.

3. Spaces of Functions and Vector Fields

To state our results various spaces of functions and vector fields are defined. A listing of spaces and projections is provided for the convenience of readers in an appendix near the end of this paper. The space $L^p(\Omega)$ with $1 \le p < \infty$, is the usual space of (equivalence classes of) Lebesgue measurable functions $\varphi : \Omega \to \mathbb{R}$ for which $|\varphi|^p$ is Lebesgue integrable on Ω. They are Banach spaces under the norm

$$\|\varphi\|_p^p := \int_\Omega |\varphi(x)|^p \; d^3x.$$

When $p = 2$, these are real Hilbert spaces, the inner product is denoted $\langle u, v\rangle$ and the subscript on the norm is omitted. All integrals are Lebesgue integrals and d^3x denotes integration with respect to 3-dimensional Lebesgue measure. When the domain of integration is omitted it is assumed to be Ω. The function φ is in $L^p_{\text{loc}}(\Omega)$ if $\varphi \in L^p(K)$ for every compact subset K of Ω.

Let $W^{1,p}(\Omega)$ be the usual real Sobolev space of functions. Derivatives are usually taken in the weak sense and the j^{th} weak derivative of $\varphi(x)$ is written $\varphi_{,j}(x) := \partial\varphi(x)/\partial x_j$.

When $\varphi \in W^{1,p}(\Omega)$ for some $p \geq 1$ and Ω obeys (B1), then the *trace of* φ *on* $\partial\Omega$ is well-defined and is a Lebesgue integrable function, see [11, Section 4.2] for details. Surface integrals are denoted by $d\sigma$ and are defined using 2-dimensional Hausdorff measure.

If $\varphi,\ \psi \in W^{1,p}(\Omega)$ for some $p \geq 3/2$, then the Gauss-Green Theorem holds in the form

$$\int_\Omega \varphi\, \psi_{,i}\, d^3x \;=\; \int_{\partial\Omega} \varphi\, \psi\, \nu_i\, d\sigma - \int_\Omega \psi\, \varphi_{,i}\, d^3x \tag{3.1}$$

for $i \in \{1, 2, 3\}$.

When $v : \Omega \to \mathbb{R}^3$ is a Lebesgue measurable vector field, its Cartesian components are denoted v_i, so that $v(x) = (v_1(x), v_2(x), v_3(x))$. When v is weakly differentiable, its derivative matrix is $Dv(x) := (v_{j,k}(x))$. v is said to be in $L^2(\Omega;\mathbb{R}^3)$, or $H^1(\Omega;\mathbb{R}^3)$, when each component v_j is in $L^2(\Omega)$, or $H^1(\Omega)$, respectively. These are Hilbert spaces with respect to the inner products

$$\langle u, v\rangle \;:=\; \int_\Omega u(x)\cdot v(x)\; d^3x, \qquad \text{and} \tag{3.2}$$

$$\langle u, v\rangle_1 \;:=\; \int_\Omega [\; u(x)\cdot v(x) \;+\; \sum_{j,k=1}^{3} u_{j,k}(x)\; v_{j,k}(x)\;]\; d^3x. \tag{3.3}$$

The corresponding norms are denoted $\|u\|, \|u\|_1$ respectively. When no subscript is indicated, the corresponding norm is an L^2-norm. The field is in $L^p(\Omega;\mathbb{R}^3)$ for $1 \leq p \leq \infty$ provided

$$\|v\|_p^p \;:=\; \int_\Omega |v(x)|^p\; d^3x \;<\; \infty.$$

The class of all such fields is a real Banach space with this norm.

When $\varphi : \Omega \to \mathbb{R}$ is in $W^{1,p}(\Omega)$ then its *gradient* is the vector field $\nabla\varphi : \Omega \to \mathbb{R}^3$ defined by

$$\nabla\varphi(x) := \big(\varphi_{,1}(x), \varphi_{,2}(x), \varphi_{,3}(x)\big) \tag{3.4}$$

and this field is in $L^p(\Omega;\mathbb{R}^3)$. When $v : \Omega \to \mathbb{R}^3$ is a C^1 vector field, its (classical) *divergence* is the function defined by

$$\operatorname{div} v(x) \;:=\; v_{1,1}(x) + v_{2,2}(x) + v_{3,3}(x) \tag{3.5}$$

The *curl* of v is the vector field defined by

$$\operatorname{curl} v(x) \;:=\; \big(v_{3,2} - v_{2,3}, v_{1,3} - v_{3,1}, v_{2,1} - v_{1,2}\big). \tag{3.6}$$

where $v_{i,j}$ is the classical j^{th} derivative of v_i.

Let $C_c^\infty(\Omega)$ be the space of all C^∞ functions on Ω with compact support. When $v \in L^p(\Omega;\mathbb{R}^3)$ for some $p \geq 1$, then a function $\rho \in L^1_{\text{loc}}(\Omega)$ is the (weak) *divergence* of v on Ω provided

$$\int_\Omega [\varphi\, \rho + \nabla\varphi\cdot v]\; d^3x \;=\; 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3.7}$$

The field is *solenoidal* on Ω if this holds with $\rho \equiv 0$ on Ω. Similarly when $v \in L^p(\Omega;\mathbb{R}^3)$ for some $p \geq 1$, then a field $\omega \in L^1_{\text{loc}}(\Omega;\mathbb{R}^3)$ is the (weak) *curl of* v *on* Ω provided

$$\int_\Omega [v \cdot \operatorname{curl} z - \omega \cdot z]\ d^3x \ = \ 0 \qquad \text{for all } z \in C_c^\infty(\Omega;\mathbb{R}^3). \tag{3.8}$$

v is *irrotational on* Ω provided this holds with $\omega \equiv 0$.

The field v is said to be in $H(\operatorname{div},\Omega)$ provided $v \in L^2(\Omega;\mathbb{R}^3)$ and $\operatorname{div} v \in L^2(\Omega)$. The space of all such fields is a real Hilbert space with respect to the inner product

$$\langle u, v\rangle_d \ := \ \int_\Omega \ [u \cdot v \ + \ \operatorname{div} u. \operatorname{div} v]\ d^3x. \tag{3.9}$$

Similarly $H(\operatorname{curl},\Omega)$ is the space of all fields $v \in L^2(\Omega;\mathbb{R}^3)$ such that $\operatorname{curl} v$ is also in $L^2(\Omega;\mathbb{R}^3)$. It is a real Hilbert space with respect to the inner product

$$\langle u, v\rangle_c \ := \ \int_\Omega \ [u \cdot v \ + \ \operatorname{curl} u \cdot \operatorname{curl} v]\ d^3x. \tag{3.10}$$

Let

$$H_{DC}(\Omega) = \{v \in L^2(\Omega;\mathbb{R}^3) : \operatorname{div} v \in L^2(\Omega) \text{ and } \operatorname{curl} v \in L^2(\Omega;\mathbb{R}^3)\}. \tag{3.11}$$

This is a Hilbert space under the inner product

$$\langle u, v\rangle_{DC} \ := \ \int_\Omega [u \cdot v \ + \ \operatorname{curl} u \cdot \operatorname{curl} v \ + \ \operatorname{div} u. \operatorname{div} v]\ d^3x, \tag{3.12}$$

which is called the *DC-inner product*. Further discussion of these spaces may be found in Girault and Raviart [14, Chapter 1] or Cessenat [9, Chapter IX, part A].

A number of results about the boundary behavior of vector fields are required. Let $C(\overline{\Omega} : \mathbb{R}^3)$ be the space of all continuous vector fields on $\overline{\Omega}$. Given $v \in C(\overline{\Omega} : \mathbb{R}^3)$, the normal component of v on the boundary is the vector field

$$v_\nu(x) \ := \ (v(x) \cdot \nu(x))\ \nu(x).$$

The tangential component of the field is

$$v_\tau(x) \ := \ v(x) \ - v_\nu(x) \ = \ \nu(x) \wedge \ (v(x) \wedge \nu(x)).$$

The vector $v_\tau(x)$ is non-zero if and only if $(v(x) \wedge \nu(x)) \neq 0$.

Define the *normal trace* operator $T_\nu : C(\overline{\Omega} : \mathbb{R}^3) \cap H_{DC}(\Omega) \to C(\partial\Omega)$ by

$$T_\nu v(x) \ := \ v(x) \cdot \nu(x) \qquad \text{for } x \in \partial\Omega. \tag{3.13}$$

This operator may be extended to a continuous linear map of $H(\operatorname{div},\Omega)$ to $H^{-1/2}(\partial\Omega)$. See [9, Chapter IX, Section 1.2, Theorem 1] for the precise statement and a proof. Similarly the *tangential trace* operator $T_\tau : C(\overline{\Omega} : \mathbb{R}^3) \cap H_{DC}(\Omega) \to C(\partial\Omega : \mathbb{R}^3)$ is defined by

$$T_\tau v(x) \ := \ v(x) \wedge \nu(x) \qquad \text{for } x \in \partial\Omega. \tag{3.14}$$

This operator may be extended to a continuous linear map of $H(\operatorname{curl},\Omega)$ to $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$. See [9, Chapter IX, Section 1.2, Theorem 2] for more details.

Linearity and the Gauss-Green theorem (3.1) yield the following identities when each of the integrals are finite:

$$\int_\Omega u \cdot \nabla\varphi \, d^3x \;=\; \int_{\partial\Omega} \varphi\,(u\cdot\nu)\,d\sigma - \int_\Omega \varphi\ \operatorname{div} u\, d^3x, \tag{3.15}$$

$$\int_\Omega u \cdot \operatorname{curl} v\, d^3x \;=\; \int_{\partial\Omega} v\cdot(u\wedge\nu)\,d\sigma \;+\; \int_\Omega v\cdot\operatorname{curl} u\, d^3x. \tag{3.16}$$

Sometimes in treating line and surface integrals on smooth subsets of $\overline{\Omega}$ we use the notation of differential forms and omit the symbol for the measure.

When $u \in H(\operatorname{div},\Omega), \varphi \in H^1(\Omega)$, (3.15) still holds with the surface integral replaced by the pairing of $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Similarly when $u \in H(\operatorname{curl},\Omega), v \in H^1(\Omega;\mathbb{R}^3)$, (3.16) remains valid with the surface integral replaced by the pairing of $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$.

These formulae permit us to define the normal and tangential traces of general fields on Ω. When $u \in L^1(\Omega;\mathbb{R}^3), \operatorname{div} u \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ we say that $u\cdot\nu := g$ on $\partial\Omega$ provided

$$\int_\Omega [\varphi \operatorname{div} u + \nabla\varphi\cdot u]\ d^3x \;=\; \int_{\partial\Omega} g\,\varphi\, d\sigma. \tag{3.17}$$

for all $\varphi \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. This definition ensures that (3.15) holds for the fields used here.

When $u \in L^1(\Omega;\mathbb{R}^3), \operatorname{curl} u \in L^1(\Omega;\mathbb{R}^3)$ and τ is a L^1-vector field on $(\partial\Omega, d\sigma)$, we say that $u\wedge\nu \;=\; \tau$ on $\partial\Omega$ provided $\tau\cdot\nu = 0, \quad \sigma$ *a.e.* on $\partial\Omega$ and

$$\int_\Omega [u\cdot\operatorname{curl} v - v\cdot\operatorname{curl} u]\ d^3x \;=\; \int_{\partial\Omega} v\cdot\tau\, d\sigma \tag{3.18}$$

for all $v \in W^{1,\infty}(\Omega;\mathbb{R}^3) \cap C(\overline{\Omega}:\mathbb{R}^3)$.

The spaces $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ and $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ are the closed subspaces of fields $u \in H^1(\Omega;\mathbb{R}^3)$ which satisfy, in the trace sense,

$$u\cdot\nu \;=\; 0 \quad \text{or} \quad u\wedge\nu \;=\; 0 \text{ on } \partial\Omega, \text{ respectively.} \tag{3.19}$$

Friedrichs showed in [13] that these spaces can be identified with the subspaces of $H_{DC}(\Omega)$ that obey the same boundary conditions; see also [5] for a recent discussion of these issues.

4. Weighted Orthogonal Decompositions

To describe the solvability of boundary value problems for *div-curl* systems of the form (2.1)–(2.2), we use some special representations of the solution. We seek solutions of the form

$$u(x) \;=\; \nabla\varphi(x) + \varepsilon(x)^{-1}\operatorname{curl} A(x) + h(x). \tag{4.1}$$

Here φ is called a *scalar potential* for the field v, A is called the *vector potential* for v and h is an ε-harmonic vector field on Ω. There are many possible such decompositions; here we apply two such decompositions defined by orthogonal projections on $L^2(\Omega;\mathbb{R}^3)$ with respect to the weighted inner product

$$\langle u, v\rangle_\varepsilon := \int_\Omega \big(\varepsilon(x)u(x)\big)\cdot v(x)\, d^3x. \tag{4.2}$$

The particular choices to be used here result in the decoupling of the equations for φ, A and h and permit a weak interpretation of the boundary conditions.

When ε satisfies (E1), the inner product (4.2) and concomitant norm are equivalent to the usual ones on $L^2(\Omega;\mathbb{R}^3)$. Two subspaces V, W of $L^2(\Omega;\mathbb{R}^3)$ are said to be *ε-orthogonal* when they are orthogonal with respect to the inner product (4.2). A field $v \in L^2(\Omega;\mathbb{R}^3)$ is said to be *ε-solenoidal* on Ω provided

$$\int_\Omega [\ (\varepsilon v)\cdot\nabla\varphi\ ]\ d^3x \ =\ 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{4.3}$$

This is the weak version of the equation

$$\operatorname{div}\big(\varepsilon(x)v(x)\big) \ =\ 0 \qquad \text{on } \Omega. \tag{4.4}$$

Define

$$G(\Omega) := \{\nabla\varphi\colon \varphi \in H^1(\Omega)\}, \text{ and } G_0(\Omega) := \{\nabla\varphi\colon \varphi \in H_0^1(\Omega)\}.$$

These are subspaces of $L^2(\Omega;\mathbb{R}^3)$ and their ε-orthogonal complements may be characterized as follows.

Proposition 4.1. *Assume (B1) and (E1) hold and $v \in L^2(\Omega;\mathbb{R}^3)$. Then v is*

(i) *ε-orthogonal to $G_0(\Omega)$ if and only if v is ε-solenoidal,*
(ii) *ε-orthogonal to $G(\Omega)$ if and only if v is ε-solenoidal and satisfies (in the sense of (3.17))*

$$\varepsilon v\cdot\nu \ =\ 0 \qquad \text{on } \partial\Omega. \tag{4.5}$$

Proof. $v \in L^2(\Omega;\mathbb{R}^3)$ is ε-orthogonal to $G_0(\Omega)$ if and only if (4.3) holds for all $\varphi \in H_0^1(\Omega)$. Thus (i) holds as $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. When v is ε-orthogonal to $G(\Omega)$, then (i) holds so εv is in $H(\mathrm{div},\Omega)$ and v is ε-solenoidal. This implies that $\varepsilon v\cdot\nu$ is in $H^{-1/2}(\partial\Omega)$. Substitute εv for u in (3.17), then v is ε-orthogonal to $G(\Omega)$ if and only if (4.5) holds. □

Define the spaces $V_\varepsilon(\Omega)$ and $V_{\varepsilon\nu 0}(\Omega)$ to be the ε-orthogonal complements of $G_0(\Omega)$ and $G(\Omega)$ respectively in $L^2(\Omega;\mathbb{R}^3)$. Then

$$L^2(\Omega;\mathbb{R}^3) \ =\ G_0(\Omega)\oplus_\varepsilon V_\varepsilon(\Omega) \ =\ G(\Omega)\oplus_\varepsilon V_{\varepsilon\nu 0}(\Omega). \tag{4.6}$$

Here $\oplus_\varepsilon$ represents the orthogonal direct sum with the inner product (4.2). Part (i) of Proposition 4.1 may be interpreted as saying that an L^2 field satisfies (4.3) if and only if it is in $V_\varepsilon(\Omega)$. Similarly $v \in L^2(\Omega;\mathbb{R}^3)$ is a weak solution of (4.4) and (4.5) if and only if it is in $V_{\varepsilon\nu 0}(\Omega)$.

Define

$$\operatorname{Curl}_\varepsilon(\Omega) := \{\varepsilon^{-1}\operatorname{curl} A : A \in H^1(\Omega;\mathbb{R}^3)\}, \tag{4.7}$$

$$\operatorname{Curl}_{\varepsilon\nu 0}(\Omega) := \{\varepsilon^{-1}\operatorname{curl} A : A \in H^1_{\nu 0}(\Omega;\mathbb{R}^3)\}, \quad \text{and} \tag{4.8}$$

$$\operatorname{Curl}_{\varepsilon\tau 0}(\Omega) := \{\varepsilon^{-1}\operatorname{curl} A : A \in H^1_{\tau 0}(\Omega;\mathbb{R}^3)\}. \tag{4.9}$$

When $\varepsilon(x) \equiv I_3$ on Ω, the subscript ε is omitted in these definitions. When ε satisfies (E1), these are subspaces of $L^2(\Omega;\mathbb{R}^3)$ and their ε-orthogonal complements may be characterized as follows.

Proposition 4.2. *Assume (B1) and (E1) hold, then* $\operatorname{Curl}_\varepsilon(\Omega) = \operatorname{Curl}_{\varepsilon\nu 0}(\Omega)$. *If* $v \in L^2(\Omega;\mathbb{R}^3)$ *then* v *is*

(i) *ε-orthogonal to* $\operatorname{Curl}_{\varepsilon\tau 0}(\Omega)$ *if and only if v is irrotational on Ω,*
(ii) *ε-orthogonal to* $\operatorname{Curl}_\varepsilon(\Omega)$ *if and only if v is irrotational and satisfies, in the sense of (3.18),*

$$v \wedge \nu = 0 \quad \text{on } \partial\Omega. \tag{4.10}$$

Proof. Suppose $B = \varepsilon^{-1}\operatorname{curl} A \in \operatorname{Curl}_\varepsilon(\Omega)$. Then $A \in H^1(\Omega;\mathbb{R}^3)$ and there is a weak solution $\varphi \in H^1(\Omega)$ of the Neumann problem

$$\Delta\varphi = \operatorname{div} A \quad \text{on } \Omega \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu} = A\cdot\nu \quad \text{on } \partial\Omega. \tag{4.11}$$

See Theorem 6.1 below for a proof of a more general result. Define $C := A - \nabla\varphi$. Then C is solenoidal, with $\operatorname{curl} C = \operatorname{curl} A$ and $C \cdot \nu = 0$ on $\partial\Omega$. Thus $C \in H_{DC\nu 0}(\Omega)$, so it is in $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ from [9, Chapter IX, Section 1.2, Theorem 3]. Also $B = \varepsilon^{-1}\operatorname{curl} C$ so $B \in \operatorname{Curl}_{\varepsilon\nu 0}(\Omega)$ and the first claim holds.

(i) From (4.2), $v \in L^2(\Omega;\mathbb{R}^3)$ is ε-orthogonal to $\operatorname{Curl}_{\varepsilon\tau 0}(\Omega)$ if and only if

$$\int_\Omega [v \cdot \operatorname{curl} A]\, d^3x = 0 \quad \text{for all } A \in H^1_{\tau 0}(\Omega;\mathbb{R}^3). \tag{4.12}$$

Since $C_c^\infty(\Omega;\mathbb{R}^3)$ is a subspace of $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$, this implies that v is irrotational on Ω so it is in $H(\operatorname{curl},\Omega)$. Conversely if v is irrotational on Ω, use (3.18) with $u \in H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ to see that it is ε-orthogonal to $\operatorname{Curl}_{\varepsilon\tau 0}(\Omega)$.

(ii) If v is ε-orthogonal to $\operatorname{Curl}_\varepsilon(\Omega)$, it is irrotational on Ω from part (i). Substitute in (3.18); then (4.10) holds. Conversely, when $v \in H^1(\Omega;\mathbb{R}^3)$ and is irrotational, (4.10) and (3.18) implies that v is ε-orthogonal to $\operatorname{Curl}_\varepsilon(\Omega)$. By density, this extends to $L^2(\Omega;\mathbb{R}^3)$. □

It is worth noting that parts (ii) of the above results imply that L^2-fields on Ω which obey an L^2-orthogonality condition not only are weak solutions of an equation on Ω, but also satisfy specific weak boundary conditions on $\partial\Omega$.

5. Harmonic and ε-harmonic Vector Fields

The last result shows that the irrotational vector fields on Ω may be characterized as the ε-orthogonal complements of specific subspaces. It is natural to ask if $G(\Omega)$ and $G_0(\Omega)$ are these spaces? As seen later, the answer depends on the topology of Ω.

The field $v \in L^2(\Omega;\mathbb{R}^3)$ is *ε-harmonic*, (respectively *harmonic*) on Ω if it is ε-solenoidal, (respectively solenoidal) and irrotational on Ω. Define $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ and $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ to be the spaces of all L^2 vector fields on Ω which are ε-orthogonal to $G(\Omega) \oplus_\varepsilon \mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$, and $G_0(\Omega) \oplus_\varepsilon \mathrm{Curl}_{\varepsilon\nu 0}(\Omega)$, respectively. That is $h \in \mathcal{H}_{\varepsilon\nu 0}(\Omega)$ provided it is in $L^2(\Omega;\mathbb{R}^3)$ and

$$\int_\Omega (\varepsilon h)\cdot\nabla\varphi\, d^3x \;=\; 0 \qquad \text{and} \qquad \int_\Omega h\cdot \mathrm{curl}\, A\, d^3x \;=\; 0 \tag{5.1}$$

for all $\varphi \in G(\Omega), A \in \mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$. Similarly h is in $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ provided (5.1) holds for all $\varphi \in G_0(\Omega), A \in \mathrm{Curl}_{\varepsilon\nu 0}(\Omega)$. These are weak forms of the system

$$\mathrm{div}\,\varepsilon h \;=\; 0 \qquad \text{and} \qquad \mathrm{curl}\, h \;=\; 0 \quad \text{on } \Omega \tag{5.2}$$

subject to further boundary conditions associated with these orthogonality conditions. This may be summarized as follows.

Proposition 5.1. *Assume (B1) and (E1) hold, then a vector field $h \in L^2(\Omega;\mathbb{R}^3)$ is in*

(i) $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ *if and only if h is ε-harmonic on Ω and satisfies (4.5),*
(ii) $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ *if and only if h is ε-harmonic on Ω and satisfies (4.10).*

When $\varepsilon = I_3$ and h is in either $\mathcal{H}_{\nu 0}(\Omega)$ or $\mathcal{H}_{\tau 0}(\Omega)$, then $h \in H^1(\Omega;\mathbb{R}^3)$.

Proof. The first two parts follows from the last two propositions of Section 4. When $\varepsilon = I_3$, the fact that any such field is actually H^1 is a consequence of [9, Chapter 9, Section 1.2, Theorem 3]. □

This result may be refined to the following *weighted Hodge-Weyl* theorem which justifies the representation (4.1).

Theorem 5.2. *Assume that (B1) and (E1) hold, then*

$$L^2(\Omega;\mathbb{R}^3) \;=\; G(\Omega) \oplus_\varepsilon \mathrm{Curl}_{\varepsilon\tau 0}(\Omega) \oplus_\varepsilon \mathcal{H}_{\varepsilon\nu 0}(\Omega), \tag{5.3}$$

$$L^2(\Omega;\mathbb{R}^3) \;=\; G_0(\Omega) \oplus_\varepsilon \mathrm{Curl}_{\varepsilon}(\Omega) \oplus_\varepsilon \mathcal{H}_{\varepsilon\tau 0}(\Omega), \tag{5.4}$$

and the spaces $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ and $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ are finite dimensional.

Proof. Here the direct sums are with respect to the weighted inner product (4.2). These decompositions follow from Proposition 5.1, provided that each of the relevant spaces is closed. These closure properties are proved in Corollaries 6.2, 6.4, 8.2 and 8.4 in the next three sections. The explicit characterization and dimension properties of the subspaces of ε-harmonic fields are described in Sections 10 and 11. □

When $\varepsilon(x) \equiv I_3$, the representation (4.1) in this theorem has been called a *Hodge-Weyl* decomposition. Hodge originally studied such formulae on quite general manifolds, see Abraham, Marsden & Ratiu [1, Section 7.5] for such a result. Weyl, in [19], used singular integral operator methods to prove similar results when Ω is a region in $\mathbb{R}^3$ and these results were proved using functional-analytic methods by Cessenat in [9, Chapter IX, Section 1.3]. In the next few sections, the methodology of Auchmuty [3] is generalized to prove this weighted version of the decomposition theorem.

6. Characterization of the Scalar Potential

In this section, the scalar potentials in Theorem 5.2 are described explicitly. If H is a Hilbert space, the projection of H onto a closed subspace V can be characterized variationally by Riesz' Theorem. When $u \in H$, its projection $\hat{u} := \mathcal{P}_V u$ onto V is the minimizer of $\|u - v\|$ over V. Let $H = L^2(\Omega; \mathbb{R}^3)$ with the weighted inner product (4.2), let V be the closure of $G(\Omega)$ and $u \in L^2(\Omega; \mathbb{R}^3)$. Then the scalar potential φ in the representation (5.3) minimizes the functional $\mathcal{D}_u : H^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}_u(\varphi) := \int_\Omega [(\varepsilon(\nabla\varphi) \cdot \nabla\varphi) - 2(\varepsilon u) \cdot \nabla\varphi] \, d^3x. \tag{6.1}$$

Let $H^1_m(\Omega)$ be the subspace of $H^1(\Omega)$ of functions with zero mean value. It is a closed subspace and

$$\mathcal{D}_u(\mathcal{P}_m\varphi) = \mathcal{D}_u(\varphi) \quad \text{for all } \varphi \in H^1(\Omega), \tag{6.2}$$

where $\mathcal{P}_m\varphi := \varphi - \overline{\varphi}$ is the projection of $H^1(\Omega)$ onto $H^1_m(\Omega)$ and $\overline{\varphi}$ is the mean value of φ on Ω. The basic results about the minimization of $\mathcal{D}_u$ may be summarized as follows.

Theorem 6.1. *Assume Ω satisfies (B1) and ε satisfies (E1). For each $u \in L^2(\Omega; \mathbb{R}^3)$, there is a unique function $\varphi_u \in H^1_m(\Omega)$ which minimizes $\mathcal{D}_u$ on $H^1_m(\Omega)$. Moreover φ_u is the unique solution in $H^1_m(\Omega)$ of*

$$\int_\Omega \varepsilon(\nabla\varphi - u) \cdot \nabla\psi \, d^3x = 0 \quad \textit{for all } \psi \in H^1_m(\Omega). \tag{6.3}$$

A function $\varphi \in H^1(\Omega)$ minimizes $\mathcal{D}_u$ on $H^1(\Omega)$ if and only if φ is a solution of (6.3).

Proof. The functional $\mathcal{D}_u$ defined by (6.1) is continuous and convex by standard arguments. Hence it is weakly lower semi-continuous on $H^1(\Omega)$ and on $H^1_m(\Omega)$. Then (2.3) and Poincaré's inequality (see [2]) imply there is a $c_1(\Omega) > 0$ such that

$$\int_\Omega (\varepsilon(\nabla\varphi) \cdot \nabla\varphi) \, d^3x \geq e_0 \int_\Omega |\nabla\varphi(x)|^2 \, d^3x \geq e_0 c_1(\Omega) \int_\Omega |\varphi(x)|^2 \, d^3x \tag{6.4}$$

for all $\varphi \in H^1_m(\Omega)$. Thus

$$\mathcal{D}_u(\varphi) \geq \frac{1}{2} \int_\Omega e_0 \left[|\nabla\varphi|^2 + c_1(\Omega)\varphi^2 \right] \, d^3x - e_1 \|u\| \|\nabla\varphi\|.$$

Since $c_1(\Omega)$ is positive, this is coercive and strictly convex on $H^1_m(\Omega)$, so there is a unique minimizer φ_u of $\mathcal{D}_u$ on $H^1_m(\Omega)$. See [20, Chapter 42] or [7, Chapter 6] for these existence theorems. The functional $\mathcal{D}_u$ is Gateaux differentiable on $H^1(\Omega)$ and its derivative $\mathcal{D}'_u(\varphi)$ satisfies

$$\langle \mathcal{D}'_u(\varphi), \psi\rangle \;=\; 2\int_\Omega \varepsilon(\nabla\varphi - u)\cdot\nabla\psi\; d^3x \tag{6.5}$$

for all $\psi \in H^1(\Omega)$. If φ minimizes $\mathcal{D}_u$ on $H^1(\Omega)$, then $\langle \mathcal{D}'_u(\varphi), \psi\rangle = 0$ for all $\psi \in H^1(\Omega)$ and thus (6.3) holds. Thus $\varphi_u = \mathcal{P}_m\varphi$ satisfies (6.3) as it only differs from φ by a constant. □

When ε satisfies (E1), and εu is a sufficiently nice field on $\overline{\Omega}$, then (6.3) is the weak form of the elliptic boundary value problem

$$\operatorname{div}(\varepsilon\nabla\varphi) \;=\; \operatorname{div}(\varepsilon u) \text{ on } \Omega, \text{ and} \tag{6.6}$$

$$(\varepsilon\nabla\varphi)\cdot\nu \;=\; \varepsilon u\cdot\nu \text{ on } \partial\Omega. \tag{6.7}$$

In this case, the scalar potentials φ_u in (5.3) may be taken to be any H^1-weak solution of this boundary-value problem.

Corollary 6.2. *With Ω, ε as above, the space $G(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^3)$. If $u \in L^2(\Omega;\mathbb{R}^3)$, the ε-orthogonal projection of u onto $G(\Omega)$ is $\nabla\varphi$ where φ is any solution of (6.3).*

Proof. Take $H = L^2(\Omega;\mathbb{R}^3)$, $V = G(\Omega)$ in [3, Corollary 3.3]. Since there is a minimizer of $\mathcal{D}_u$ on $H^1(\Omega)$ for each $u \in H$, the space $G(\Omega)$ is a closed subspace of H and the projection of u into $G(\Omega)$ is $\nabla\varphi$ where φ is any solution of (6.3). □

The projection of $L^2(\Omega;\mathbb{R}^3)$ onto $G_0(\Omega)$ may be analyzed in a similar manner. The functionals are the same but the space of allowable potentials is $H^1_0(\Omega)$. This leads to the following.

Theorem 6.3. *Let Ω be a bounded, open set in $\mathbb{R}^3$ and suppose ε satisfies (E1). For each $u \in L^2(\Omega;\mathbb{R}^3)$, there is a unique function φ_0 that minimizes $\mathcal{D}_u$ on $H^1_0(\Omega)$. Moreover φ_0 is the unique solution in $H^1_0(\Omega)$ of*

$$\int_\Omega \varepsilon(\nabla\varphi - u)\cdot\nabla\psi\; dx \;=\; 0, \;\textit{for all } \psi \in H^1_0(\Omega). \tag{6.8}$$

Proof. The crucial step here is the verification of the Poincaré inequality on $H^1_0(\Omega)$. This is a standard result (see [2]). The rest of the proof parallels that of Theorem 6.1. □

Corollary 6.4. *When ε satisfies (E1) and Ω is a bounded open set in $\mathbb{R}^3$, $G_0(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^3)$. If $u \in L^2(\Omega;\mathbb{R}^3)$, the ε-orthogonal projection of u onto $G_0(\Omega)$ is $\nabla\varphi_0$ where φ_0 is the unique solution of (6.8).*

Proof. Take $V = G_0(\Omega)$ and repeat the arguments in the proof of Corollary 6.2. □

Essentially, this projection is $\nabla\varphi_0$ where φ_0 is a weak solution in $H^1_0(\Omega)$ of equation (6.6).

7. Vector Potential of Sobolev Fields

In this section some further properties of certain potential representations of fields in (subspaces of) $H^1(\Omega;\mathbb{R}^3)$ are described. In particular we use the results to specify some uniqueness results for the vector potential of such fields.

Let $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$, $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ be the spaces defined at the end of Section 3, and $V^1_{\nu 0}(\Omega)$, $V^1_{\tau 0}(\Omega)$ be their subspaces of solenoidal fields. Throughout this section we use results from the previous section with $\varepsilon(x) \equiv I_3$ on Ω. The following result may be compared with Proposition 4.1.

Proposition 7.1. *Assume Ω satisfies (B1). Given $u \in H^1_{\tau 0}(\Omega;\mathbb{R}^3)$, there is a unique $v \in V^1_{\tau 0}(\Omega)$ and $\varphi_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ such that*

$$u \;=\; \nabla\varphi_0 + v \tag{7.1}$$

and the fields $\nabla\varphi_0$, v are L^2-orthogonal.

Proof. Let φ_0 be a minimizer of $\mathcal{D}_u$ on $H^1_0(\Omega)$. It exists and is unique from Theorem 6.3. Condition (B1) and elliptic regularity results imply that φ_0 is in $H^2(\Omega)$, see [10, Section 6.3, Theorem 4] for a proof. Define $v := u - \nabla\varphi_0$, then L^2-orthogonality follows from (6.8) and $\varepsilon = I_3$. Also $\langle \operatorname{curl} A, \nabla\varphi_0\rangle = 0$ for all $A \in H^1(\Omega;\mathbb{R}^3)$ [3, Theorem 3.1 (II)] or [9, Chapter IX, Section 1.3, Proposition 3]. Then (3.18) implies that $(\nabla\varphi_0)\wedge\nu \;=\; 0$ on $\partial\Omega$ in a weak sense. Thus $v \in V^1_{\tau 0}(\Omega)$ as claimed. □

The map $\mathcal{P}_{G0} : H^1(\Omega;\mathbb{R}^3) \to H^1(\Omega;\mathbb{R}^3)$ defined by $\mathcal{P}_G u := \nabla\varphi_0$ with φ_0 as in this proof is a continuous linear projection when (B1) holds. Thus its complement $\mathcal{P}_S u := u - \nabla\varphi_0$ is the projection onto $V^1_{\tau 0}(\Omega)$. The following is a similar decomposition result for $H^1(\Omega;\mathbb{R}^3)$. It is worth noting that the regularity of φ here is a consequence of results about vector fields.

Proposition 7.2. *Assume Ω satisfies (B1). Given $u \in H^1(\Omega;\mathbb{R}^3)$, there is a unique $v \in V^1_{\nu 0}(\Omega)$ and $\varphi \in H^2(\Omega) \cap H^1_m(\Omega)$ such that*

$$u \;=\; \nabla\varphi + v \tag{7.2}$$

and the fields $\nabla\varphi, v$ are L^2-orthogonal.

Proof. Let $\mathcal{D}_u$ be the functional defined by (6.2) with $\varepsilon \equiv I_3$ on Ω and φ be a minimizer of $\mathcal{D}_u$ on $H^1_m(\Omega)$. It exists and is a solution of (6.3), so it is a weak solution of $\Delta\varphi = \operatorname{div} u \in L^2(\Omega)$. Thus $g := \nabla\varphi$ is in $H_{DC}(\Omega)$ as $\operatorname{curl} g = 0$ in a weak sense. Define $v := u - \nabla\varphi$, then $v \in V^1_{\nu 0}(\Omega)$ so it is in $H_{DC\nu 0}(\Omega)$. From [9, Chapter IX, Section 1.2, Theorem 3], $v \in H^1(\Omega;\mathbb{R}^3)$ so g is also and thus $\varphi \in H^2(\Omega)$ as claimed. The orthogonality follows from (6.3) with $\varepsilon = I_3$. □

Just as above, when (B1) holds, the map $\mathcal{P}_G : H^1(\Omega;\mathbb{R}^3) \to H^1(\Omega;\mathbb{R}^3)$ defined by $\mathcal{P}_G u := \nabla\varphi$ with φ as in this proof is a continuous linear projection and its complement is the projection onto $V^1_{\nu 0}(\Omega)$.

We next characterize the vector potentials of the solenoidal components of these Sobolev fields. This depends on the topology of the region Ω. Let $\mathcal{H}_{\nu 0}(\Omega), \mathcal{H}_{\tau 0}(\Omega)$ be the spaces of harmonic fields on Ω which satisfy the boundary conditions (3.19) respectively. It is shown in [3, Sections 9, 10] that these are finite dimensional subspaces of $H^1(\Omega;\mathbb{R}^3)$ when (B1) holds. There they are characterized as the zero-eigenspaces of a vector-valued eigenproblem.

Set dim $\mathcal{H}_{\nu 0}(\Omega) = L$ and dim$\mathcal{H}_{\tau 0}(\Omega) = J$. Then de Rham theory says that the first and second Betti numbers of Ω are L and J respectively. Let $\mathcal{P}_{H1}$ and $\mathcal{P}_{H2}$ be the projections of $H^1(\Omega;\mathbb{R}^3)$ onto these spaces. $\mathcal{P}_{H1} = 0$ if Ω is simply connected and $\mathcal{P}_{H2} = 0$ if $\partial\Omega$ is connected. Let $Z_{\nu 0}(\Omega)$ be the subspace of $V^1_{\nu 0}(\Omega)$ which is L^2-orthogonal to $\mathcal{H}_{\nu 0}(\Omega)$, and define $\mathcal{P}_\nu : H^1(\Omega;\mathbb{R}^3) \to Z_{\nu 0}(\Omega)$ by

$$\mathcal{P}_\nu u \ := \ u - \ \mathcal{P}_G u - \mathcal{P}_{H1} u \tag{7.3}$$

where $\mathcal{P}_G$ and $\mathcal{P}_{H1}$ are defined as above. From these definitions $\hat{u}$ differs from u by an irrotational field, so $\operatorname{curl} u = \operatorname{curl}\hat{u}$. The following result is [3, Theorem 5.1].

Theorem 7.3. *Assume Ω satisfies (B1) and $\mathcal{P}_\nu$ is defined by (7.3). For each $u \in H^1(\Omega;\mathbb{R}^3)$, $\hat{u} := \mathcal{P}_\nu u$ satisfies $\operatorname{curl} u = \operatorname{curl}\hat{u}$ and there is a constant $c > 0$ such that*

$$\int_\Omega |\operatorname{curl} u|^2 \, d^3x \ \geq \ c \int_\Omega |\hat{u}|^2 \, d^3x, \tag{7.4}$$

There is a similar result for $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$. Define $Z_{\tau 0}(\Omega)$ be the subspace of $V^1_{\tau 0}(\Omega)$ which is L^2-orthogonal to $\mathcal{H}_{\tau 0}(\Omega)$, and define $\mathcal{P}_\tau : H^1_{\tau 0}(\Omega;\mathbb{R}^3) \to Z_{\tau 0}(\Omega)$ by

$$\mathcal{P}_\tau u \ := \ u - \mathcal{P}_{G0} u - \mathcal{P}_{H2} u \tag{7.5}$$

where $\mathcal{P}_{G0}$ is the gradient projection in Theorem 7.1 and $\mathcal{P}_{H2}$ is defined above. This holds as for the previous theorem but now (7.6) follows from [3, Theorem 6.1].

Theorem 7.4. *Assume Ω satisfies (B1) and $\mathcal{P}_\tau$ is defined by (7.5). For each $u \in H^1(\Omega;\mathbb{R}^3)$, $\tilde{u} := \mathcal{P}_\tau u$ satisfies $\operatorname{curl} u = \operatorname{curl}\tilde{u}$ and there is a constant $c > 0$ such that*

$$\int_\Omega |\operatorname{curl} u|^2 \, d^3x \ \geq \ c \int_\Omega |\tilde{u}|^2 \, d^3x. \tag{7.6}$$

It is worth noting that the optimal constants in (7.4) and (7.6) are the same and depend only on the region Ω, see [3, Section 11]. The projections $\mathcal{P}_\nu$ and $\mathcal{P}_\tau$ are used to specify unique vector potentials for fields in these Sobolev spaces.

8. Vector Potentials of L^2 fields.

For given L^2 field u, the vector potential A in (4.1) may also be defined by a variational principle. Let V be the closure of $\operatorname{Curl}_\varepsilon(\Omega)$ in $L^2(\Omega;\mathbb{R}^3)$. Given $u \in L^2(\Omega;\mathbb{R}^3)$, from Proposition 4.2, the potential A in the representation (5.4) minimizes the functional $\mathcal{C}_u : H^1_{\nu 0}(\Omega;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{C}_u(A) \ := \ \int_\Omega [(\varepsilon^{-1} \operatorname{curl} A) \cdot \operatorname{curl} A - 2u \cdot \operatorname{curl} A \,] \, d^3x, \tag{8.1}$$

Here $\varepsilon(x)^{-1}$ is the inverse matrix of $\varepsilon(x)$ and satisfies (E1) when ε does. Unfortunately this functional is not coercive on $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$. Let $Z_{\nu 0}(\Omega)$ be the space defined in the preceding section. It is a closed subspace of $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$. Let $\mathcal{P}_\nu$ be the corresponding projection onto $Z_{\nu 0}(\Omega)$ then, from Theorem 7.3,

$$\mathcal{C}_u(\mathcal{P}_\nu A) \;=\; \mathcal{C}_u(A) \quad \text{for all } A \in H^1_{\nu 0}(\Omega;\mathbb{R}^3). \tag{8.2}$$

Consider the problem of minimizing $\mathcal{C}_u$ on $Z_{\nu 0}(\Omega)$. From (8.2)

$$\inf_{A\in H^1_{\nu 0}(\Omega:\mathbb{R}^3)} \mathcal{C}_u(A) \;=\; \inf_{A\in Z_{\nu 0}(\Omega)} \mathcal{C}_u(A) \tag{8.3}$$

and $\hat{A}$ minimizes $\mathcal{C}_u$ on $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ if and only if $\mathcal{P}_\nu \hat{A}$ minimizes $\mathcal{C}_u$ on $Z_{\nu 0}(\Omega)$. The basic results about this minimization of $\mathcal{C}_u$ may be summarized as follows.

Theorem 8.1. *Assume Ω, ε satisfy (B1) and (E1). For each $u \in L^2(\Omega;\mathbb{R}^3)$, there is a unique $\hat{A}$ which minimizes $\mathcal{C}_u$ on $Z_{\nu 0}(\Omega)$. A field A minimizes $\mathcal{C}_u$ on $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ if and only if*

$$\int_\Omega \left[\varepsilon^{-1}(\operatorname{curl} A) - u\right] \cdot (\operatorname{curl} B)\, d^3x \;=\; 0 \quad \text{for all } B \in H^1_{\nu 0}(\Omega;\mathbb{R}^3). \tag{8.4}$$

In this case

$$\mathcal{P}_\nu A \;=\; \hat{A}, \quad \text{or } A \;=\; \hat{A} + \nabla\varphi + h, \tag{8.5}$$

for some $\varphi \in H^1(\Omega)$ and $h \in \mathcal{H}_{\nu 0}(\Omega)$.

Proof. When ε satisfies (E1), so does ε^{-1} and, (2.3) yields

$$e_1^{-1}\,|v|^2 \;\le\; (\varepsilon(x)^{-1}v)\cdot v \;\le\; e_0^{-1}\,|v|^2\,. \tag{8.6}$$

for each $v \in \mathbb{R}^3$. The functional $\mathcal{C}_u$ defined by (8.1) is continuous and convex on $H^1_{\nu 0}(\Omega)$, so it is weakly lower semicontinuous on $H^1_{\nu 0}(\Omega)$ and also on $Z_{\nu 0}(\Omega)$. From (8.6) and (7.4), there is a $c > 0$ such that

$$\mathcal{C}_u(A) \;\ge\; \frac{1}{2}\int_\Omega {e_1}^{-1}\left[|\operatorname{curl} A|^2 + c\,|A|^2\right]\,dx - \|v\|\,\|\operatorname{curl} A\|\,.$$

Thus $\mathcal{C}_u$ is coercive and strictly convex so there is a unique minimizer $\hat{A}$ of $\mathcal{C}_u$ on $Z_{\nu 0}(\Omega)$. Given $C \in H^1_{\nu 0}(\Omega;\mathbb{R}^3)$, there is a unique L^2-orthogonal decomposition $C \;=\; A + \nabla\varphi + h$ with $A \in H^1_{\nu 0}(\Omega;\mathbb{R}^3), \varphi \in H^1(\Omega)$ and $h \in \mathcal{H}_{\nu 0}(\Omega)$ from [3, eq. (3.8)]. In this case $\operatorname{curl} C = \operatorname{curl} A$ so (8.3) holds. Since $\hat{A}$ is the unique minimizer on $Z_{\nu 0}(\Omega)$, this representation theorem and (8.3) yields (8.5). The functional $\mathcal{C}_u$ is Gateaux differentiable on $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ and the extremality condition obeyed at a minimizer may be verified to be (8.4). □

Equation (8.4) is the weak form of the system

$$\operatorname{curl}\left(\varepsilon^{-1}\operatorname{curl}\ A\right) \;=\; \operatorname{curl} u \quad \text{on } \Omega, \tag{8.7}$$

$$(\varepsilon^{-1}\operatorname{curl}\ A)\wedge\nu \;=\; u\wedge\nu \quad \text{on } \partial\Omega, \tag{8.8}$$

$$A\cdot\nu \;=\; 0 \quad \text{on } \partial\Omega. \tag{8.9}$$

Each field A of the form (8.5) is a solution of this problem. The unique solution $\hat{A} \in Z_{\nu 0}(\Omega)$ is also solenoidal and obeys

$$\langle A, h \rangle \; = \; 0 \qquad \text{for all } h \in \mathcal{H}_{\nu 0}(\Omega). \tag{8.10}$$

Corollary 8.2. *Assume Ω, ε as above, then $\mathrm{Curl}_\varepsilon(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^3)$. If $u \in L^2(\Omega; \mathbb{R}^3)$, the ε-orthogonal projection of u onto $\mathrm{Curl}_\varepsilon(\Omega)$ is $\varepsilon^{-1} \operatorname{curl} A$ where A is any solution of (8.4).*

Proof. This proof is the same as that of Corollary 6.2 but with $\mathrm{Curl}_\varepsilon(\Omega)$ in place of $G(\Omega)$. □

The projection of $L^2(\Omega; \mathbb{R}^3)$ onto $\mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$ may be analyzed in a similar manner. The functionals are the same but the space of allowable vector potentials now is $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$. This leads to the following results whose proofs are similar to the above.

Theorem 8.3. *Let Ω and ε obey (B1) and (E1). For each $u \in L^2(\Omega; \mathbb{R}^3)$, there is a unique $\tilde{A}$ which minimizes $\mathcal{C}_u$ on $Z_{\tau 0}(\Omega)$. A field A minimizes $\mathcal{C}_u$ on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ if and only if*

$$\int_\Omega \left[\varepsilon^{-1}(\operatorname{curl} A) - u\right] \cdot (\operatorname{curl} B) \, dx \; = \; 0, \qquad \textit{for all } B \in H^1_{\tau 0}(\Omega; \mathbb{R}^3). \tag{8.11}$$

In this case

$$A \; = \; \tilde{A} + \nabla \varphi + h \qquad \textit{for some } \varphi \in H^1_0(\Omega) \textit{ and } h \in \mathcal{H}_{\tau 0}(\Omega). \tag{8.12}$$

Corollary 8.4. *When ε and Ω as above, $\mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^3)$. If $u \in L^2(\Omega; \mathbb{R}^3)$ the ε-orthogonal projection of u onto $\mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$ is $\varepsilon^{-1} \operatorname{curl} A$, where $A \in H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ is a solution of (8.11).*

Equivalently this projection is $\varepsilon^{-1} \operatorname{curl} A$ where A is a weak solution of equation (8.7) in $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$. Here the unique solution $\tilde{A}$ is solenoidal and satisfies

$$\langle A, k \rangle \; = \; 0 \qquad \text{for all } k \in \mathcal{H}_{\tau 0}(\Omega). \tag{8.13}$$

9. L^2- and ε-orthogonality

To complete the proof of Theorem 5.2, the spaces of ε-harmonic vector fields are characterized explicitly. We use the following result which may be interpreted as saying that each vector field in $L^2(\Omega; \mathbb{R}^3)$ can be written as a sum of a gradient and a curl. These results are proved in chapter IX, section 1.3 of [9] - and the versions below just extract the statements needed here.

Theorem 9.1. *Suppose Ω satisfies (B1) and $u \in L^2(\Omega; \mathbb{R}^3)$.*

(i) *If u is L^2-orthogonal to $G(\Omega)$, then $u \in Curl(\Omega)$.*
(ii) *If u is L^2-orthogonal to $Curl(\Omega)$, then $u \in G(\Omega)$.*

Proof. The proof of these follow from the analysis of [9, Chapter IX, Section 1.3]. The assumptions (B1) are sufficient to guarantee the condition (1.45) i) required there.

(i) First assume that $u \in H^1(\Omega;\mathbb{R}^3)$, then [9, Chapter IX, Section 1.3, Corollary 5] implies that

$$u \ = \ \nabla\varphi \ + \ \operatorname{curl} A \quad \text{with} \quad \operatorname{curl} A \cdot \nu = 0 \text{ on } \partial\Omega. \tag{9.1}$$

Multiply this by $\nabla\varphi$ and integrate over Ω. If u is L^2-orthogonal to $G(\Omega)$, use the divergence theorem to see that $\|\nabla\varphi\|_2^2 \ = \ 0$. Hence $u \in Curl(\Omega)$. Since (i) holds for all such H^1 fields it holds for all such L^2 fields by a density argument.

(ii) This follows from proposition 3 of section 1.3 cited above; especially with the help of Proposition 4.1 of this paper. □

A similar result holds for ε-orthogonal subspaces.

Corollary 9.2. *Suppose Ω satisfies (B1), ε satisfies (E1) and $u \in L^2(\Omega;\mathbb{R}^3)$.*

(i) *If u is ε-orthogonal to $G(\Omega)$, then $u \in \mathrm{Curl}_\varepsilon(\Omega)$,*
(ii) *If u is ε-orthogonal to $\mathrm{Curl}_\varepsilon(\Omega)$, then $u \in G(\Omega)$.*

Proof. If u is ε-orthogonal to $G(\Omega)$, then εu is orthogonal to $G(\Omega)$ so (i) follows from (i) of Theorem 9.1 and the definition (4.7). Similarly part (ii) follows directly from (ii) of Theorem 9.1. □

The question as to whether the results described in this paper hold under weaker regularity conditions on the region than (B1) appear to require that a version of this Theorem 9.1 hold on such regions. It is of considerable interest to investigate what characterization of these orthogonal complements holds under other reasonable geometric or regularity assumptions on Ω. These results are required for the explicit characterization of the ε-harmonic vector fields in the next section.

10. The space $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$

In this section the ε-harmonic vector fields in the decomposition (5.4) are characterized explicitly and we show that when Ω satisfies (B1), the dimension of the space $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ is J; one less than the number of connected components of the boundary of Ω.

Let the components of $\partial\Omega$ be denoted Σ_0, Σ_1, …, Σ_J, with the connected region Ω being a subset of the region interior to Σ_0. When $J = 0$ these spaces of harmonic vector fields are trivial. When $J \geq 1$, Ω is the region interior to Σ_0 and exterior to each of the other components Σ_j. Thus Ω may be viewed as a region with J "holes" and J is the *second Betti number* of Ω. To prove the results on the dimensions of the spaces of ε-harmonic vector fields introduced in Section 3, we obtain an explicit characterization of these fields. Assume $J \geq 1$, Ω satisfies (B1) and the Σ_j are as above. For each $1 \leq j \leq J$, let K_j be the class of functions in $H^1(\Omega)$ which satisfy

$$\chi(x) \;=\; \begin{cases} 1 & \text{for } x \in \Sigma_j, \\ 0 & \text{for } x \in \partial\Omega \setminus \Sigma_j. \end{cases} \tag{10.1}$$

It is straightforward to verify that K_j is a closed convex subset of $H^1(\Omega)$. Let $\mathcal{D}_0 : H^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$\mathcal{D}_0(\varphi) \;:=\; \int_\Omega \big(\varepsilon(\nabla\varphi) \cdot \nabla\varphi\big)\, d^2x. \tag{10.2}$$

Consider the variational problem of minimizing $\mathcal{D}_0$ on K_j. This problem has a unique minimizer χ_j and it is the unique H^1-solution in K_j of

$$\operatorname{div}\,(\varepsilon(x)\nabla\chi(x)) \;=\; 0 \quad \text{on } \Omega, \tag{10.3}$$

subject to the boundary conditions (10.1). The proof of this parallels that of Theorem 6.3. Define

$$h^{(j)}(x) \;:=\; \nabla\chi_j(x) \quad \text{for } x \in \Omega, \quad 1 \le j \le J. \tag{10.4}$$

Then each $h^{(j)}$ is a non-zero L^2-vector field on Ω. Under our regularity assumptions (B1) and (E1) on Ω, ε, each $h^{(j)}$ be continuous on $\overline{\Omega}$ and C^1 on Ω.

Theorem 10.1. *Assume Ω, ε satisfy (B1) and (E1). When $J \;=\; 0$, $\mathcal{H}_{\varepsilon\tau 0}(\Omega) = \{0\}$. When $J \;\ge\; 1$, then $\{h^{(1)}, \ldots, h^{(J)}\}$ defined by (10.4) is a basis of $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ and thus $\dim \mathcal{H}_{\varepsilon\tau 0}(\Omega) \;=\; J$.*

Proof. If $h \in \mathcal{H}_{\varepsilon\tau 0}(\Omega)$, then it is ε-orthogonal to $\mathrm{Curl}_\varepsilon(\Omega)$ by definition, so from part (ii) of Theorem 9.2, $h \;=\; \nabla\varphi_h$ on Ω, where $\nabla\varphi_h$ is the projection of h onto $G(\Omega)$ defined by Theorem 6.1 and Corollary 6.2. Now h satisfies (4.10), so

$$\nabla\varphi_h \;\wedge\; \nu \;=\; h \wedge \nu \;=\; 0 \quad \text{on } \partial\Omega.$$

Since the components of $\partial\Omega$ are C^1, there are constants κ_j with

$$\varphi_h(x) \;=\; \kappa_j \quad \text{on } \Sigma_j, \quad 0 \le j \le J.$$

When $J \;=\; 0$, this and (10.3) imply that $\varphi_h(x) \equiv \kappa_0$ is a solution for this potential and hence $h \equiv 0$, so the first claim holds. When $J \ge 1$, consider the function

$$\mathcal{X}(x) \;=\; \varphi_h(x) - \sum_{j=1}^{J} c_j \chi_j(x) - \kappa_0.$$

This function satisfies (10.3) and $\mathcal{X}(x) \;=\; \kappa_j - \kappa_0 - c_j$ on Σ_j. Take $c_j \;=\; \kappa_j - \kappa_0$ for each j. Then $\mathcal{X} \equiv 0$ on $\partial\Omega$. The only such solution of (10.3) is identically 0 on Ω, so

$$\varphi_h(x) \quad = \quad \kappa_0 + \sum_{j=1}^{J} (\kappa_j - \kappa_0)\chi_j(x) \quad \text{on } \overline{\Omega}, \text{ and} \tag{10.5}$$

$$h(x) \quad = \quad \sum_{j=1}^{J} c_j\; h^{(j)}(x), \tag{10.6}$$

upon taking gradients. Thus $\dim \mathcal{H}_{\varepsilon\tau 0}(\Omega) \le J$. If $h^{(1)}, \ldots, h^{(J)}$ are linearly dependent, there are constants $a_1, \ldots, a_J$, not all zero, such that

$$\nabla\Big(\sum_{j=1}^{J} a_j \chi_j\Big)(x) \equiv 0 \quad \text{ on } \overline{\Omega}.$$

Since Ω is connected, $\psi(x) = \sum_{j=1}^{J} a_j \chi_j(x)$ must be constant on $\overline{\Omega}$. However

$$\psi(x) = \begin{cases} a_j & \text{on } \Sigma_j,\ 1 \le j \le J, \\ 0 & \text{on } \Sigma_0. \end{cases}$$

Then $a_j = 0$ for all j which contradicts our assumption, so the theorem holds. $\square$

Suppose $J \ge 1$ and $v \in L^2(\Omega; \mathbb{R}^3)$. Then the projection of $L^2(\Omega; \mathbb{R}^3)$ onto $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ is given by

$$\mathcal{P}_{H\tau} v = \sum_{j=1}^{J} c_j h^{(j)}(x), \tag{10.7}$$

where $(c_1, \ldots, c_J)$ is the solution of the linear system

$$\sum_{j=1}^{J} h_{kj}\, c_j = v_k, \qquad 1 \le k \le J. \tag{10.8}$$

Here $h_{kj} = \langle h^{(k)}, h^{(j)} \rangle_\varepsilon$ and $v_k = \langle v, h^{(k)} \rangle_\varepsilon$. Note that this projection is orthogonal and continuous on $L^2(\Omega; \mathbb{R}^3)$ by construction and its range consists of smooth fields on Ω from regularity theory for (10.3)–(10.1). The matrix $H := (h_{jk})$ is symmetric and non-singular as it is the Grammian matrix of a set of linearly independent vector fields. In particular this projection is the zero field if and only if each $v_k = 0$. When (E1) holds, and v is smooth enough, this is the requirement that the flux of εv through each boundary component is zero. That is, for each j, $1 \le j \le J$,

$$\int_{\Sigma_j} \varepsilon v \, d\sigma = 0.$$

Let $\xi_i := \{x^{(i)}(t) : 0 \le t \le 1\}$ be a C^1 curve in $\overline{\Omega}$ with $x^{(i)}(0) \in \Sigma_0$, $x^{(i)}(1) \in \Sigma_i$ and $x^{(i)}(t) \in \Omega$ for $0 < t < 1$. When $h^{(j)}$ is defined by (10.4), for $1 \le i, j \le J$, then

$$\int_{\xi_i} h^{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases} \tag{10.9}$$

This is a consequence of the boundary condition (10.1) and the chain rule. It leads to the following characterization of coefficients in this projection.

Theorem 10.2. *Assume Ω satisfies (B1) with $J \ge 1$. If v is a continuous vector field on $\overline{\Omega}$ which is ε-orthogonal to $\mathrm{Curl}_\varepsilon(\Omega)$, the projection $\mathcal{P}_{H\tau}$ of $L^2(\Omega; \mathbb{R}^3)$ onto $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ is* *given by (10.7) with*

$$c_j \;=\; \int_{\xi_j} v, \qquad 1 \;\le\; j \;\le\; J. \tag{10.10}$$

Proof. With v as in the theorem, (5.4) implies that $v \;=\; \nabla\varphi_v + h$ for some $\varphi_v \in H_0^1(\Omega)$ and $h \in \mathcal{H}_{\varepsilon\tau 0}(\Omega)$. Since h is continuous on $\overline{\Omega}$, so is $\nabla\varphi_v$. The line integral of $\nabla\varphi_v$ over ξ_j is zero from the chain rule. Hence (10.9) and (10.6) imply (10.10). $\square$

The coefficients c_j here may be regarded as the *potential difference* of the scalar potential $\varphi_0 + \varphi_h$ of v between the boundary surfaces Σ_0 and Σ_j. For a general continuous vector field on $\overline{\Omega}$, (10.10) becomes

$$c_j \;=\; \int_{\xi_j} [v - \varepsilon^{-1}\,\mathrm{curl}\, A]$$

where A is a vector potential of v defined as in Theorem 8.1.

11. The space $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$

When $\varepsilon(x) \equiv I_3$, the space $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ arises in topology. Its construction and properties are described by Cessenat in [9, Chapter 9, Section 1.3], Foias and Temam [12], Picard [16] and Saarinen [17] amongst other references. The dimension L of this space is a topological invariant of the region. It is the *first Betti number,* $\beta_1(\Omega)$ or the *genus* of the region Ω and counts the number of *handles* in the region.

Let $\{S_1, S_2, \ldots, S_L\}$ be closed subsets of $\overline{\Omega}$ which are C^2-surfaces with boundary and satisfy

S1: $S_l \cap S_m \;=\; \emptyset$ for $1 \le \ell < m \le L$.
S2: *S_l is never tangential to $\partial\Omega$.*
S3: *the region $\Omega_c := \Omega \backslash (\cup_{\ell=1}^{L} S_\ell)$ is simply connected.*

When [S1]–[S3] hold, the sets $\{S_1, \ldots, S_L\}$ are called *cutting surfaces* for the region Ω. The above references describe the construction of a basis $\{\tilde{k}^{(\ell 1)}, \ldots, \tilde{k}^{(L)}\}$ of $\mathcal{H}_{\nu 0}(\Omega)$ by solving certain transmission problems for harmonic functions that are discontinuous across this family of cutting surfaces. Choose the fields $k^{(\ell)}$ to satisfy the flux conditions;

$$\int_{S_j} \tilde{k}^{(\ell)}\, d\sigma \;=\; \delta_{jl}, \qquad 1 \le j, \ell \le L. \tag{11.1}$$

Note that this is not the usual convention described in the above references, but it is a permissible alternative to the standard condition involving certain loop integrals.

When ε is a general matrix obeying (E1), a basis of $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ using transmission problems was described by Picard in [16] and he showed the dimension of the space is L. Also Saarinen in [17] gives an abstract proof, credited to K. J. Witsch, that under conditions similar to ours, the space $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ has dimension L. Here we first describe an explicit construction of a basis for $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ which is based on perturbation of the basis with $\varepsilon = I_3$ and does not require the solution of another transmission type problem.

Note that if a field $k \in \mathcal{H}_{\varepsilon\nu 0}(\Omega)$ then $k \in \mathrm{Curl}_{\varepsilon}(\Omega)$, from part (i) of Theorem 9.2 as k is ε-orthogonal to $G(\Omega)$. Define $h := \varepsilon k$, then $h \in V^1_{\nu 0}(\Omega)$ and is L^2-orthogonal to $G(\Omega)$. From the (unweighted) Hodge-Weyl decomposition (5.3) there is a $B \in V^1_{\tau 0}(\Omega)$ and a $\tilde{k} \in \mathcal{H}_{\nu 0}(\Omega)$ such that

$$h \;=\; \mathrm{curl}\; B + \tilde{k}. \tag{11.2}$$

Given ℓ with $1 \le \ell \le L$, define $B^{(l)}$ to be the unique minimizer of the functional $\mathcal{C}_l \colon Z_{\tau 0}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{C}_l(B) \;:=\; \mathcal{C}_{-\varepsilon^{-1}\tilde{k}^{(l)}}(B), \tag{11.3}$$

where $\mathcal{C}_u$ is defined by (8.1). This problem has a unique solution from Theorem 8.3 and $B^{(\ell)}$ satisfies (8.7) with $-\varepsilon^{-1}\tilde{k}^{(\ell)}$ in place of u. Define

$$k^{(\ell)} \;:=\; \varepsilon^{-1}[\mathrm{curl}\, B^{(\ell)} + \tilde{k}^{(\ell)}], \qquad 1 \le \ell \le L. \tag{11.4}$$

These fields are non-zero L^2 fields on Ω. They could also be defined using a modification of the proof of [9, Chapter 9, Section 1.3, Proposition 2]. These fields are continuous on $\overline{\Omega}$ and C^1 on Ω.

Theorem 11.1. *Assume Ω, ε satisfy (B1) and (E1). When Ω is simply connected, $\mathcal{H}_{\varepsilon\nu 0}(\Omega) = \{0\}$. When $L \ge 1$, then $\{k^{(1)}, \ldots, k^{(L)}\}$ defined by (11.4) is a basis of $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ and $\dim \mathcal{H}_{\varepsilon\nu 0}(\Omega) = L$.*

Proof. When $k^{(\ell)}$ is defined by (11.4), then it is irrotational, since $B^{(\ell)}$ satisfies (8.7). It is ε-solenoidal since $\tilde{k}^{(l)}$ is solenoidal. Hence it is in $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$. Integrate $\varepsilon k^{(\ell)}$ over a cutting surface S_j; then Stokes' Theorem and (11.1) yield

$$\int_{S_j} \varepsilon k^{(\ell)} \;=\; \int_{S_j} \varepsilon \tilde{k}^{(\ell)} \;=\; \delta_{j\ell}, \qquad 1 \le j, \ell \le L. \tag{11.5}$$

as each $B^{(\ell)} \in Z_{\tau 0}(\Omega)$. Hence the fields are linearly independent and are a basis as claimed. □

Suppose $L \ge 1$ and $v \in L^2(\Omega; \mathbb{R}^3)$. Then the projection of $L^2(\Omega; \mathbb{R}^3)$ onto $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ is given by

$$\mathcal{P}_{H\nu} v(x) \;=\; \sum_{\ell=1}^{L} c_\ell k^{(\ell)}(x), \tag{11.6}$$

where $(c_1, \ldots, c_L)$ is the solution of a linear system similar to (10.8). Namely

$$\sum_{\ell=1}^{L} h_{j\ell}\, c_\ell \;=\; v_j, \qquad 1 \;\le\; j \;\le\; L. \tag{11.7}$$

Here $h_{jl} = \langle k^{(j)}, k^{(l)} \rangle_\varepsilon$ and $v_j = \langle v, k^{(j)} \rangle_\varepsilon$. The matrix $H := (h_{jl})$ is symmetric and non-singular as it is the Grammian matrix of a set of linearly independent vector fields. Since the $k^{(l)}$ are smooth, the range of this projection consists of nice fields and the projection is continuous on $L^2(\Omega;\mathbb{R}^3)$ by construction. The projection is zero if and only if the flux of εv through each cutting surface is zero. That is,

$$\int_{S_\ell} \varepsilon v \, d\sigma \;=\; 0, \quad \text{for each } 1 \le \ell \le L.$$

This time the coefficients in this expansion may also be determined as fluxes across the cutting surfaces using the following result.

Theorem 11.2. *Assume Ω satisfies (B1) with $L \ge 1$. If v is a continuous vector field on $\overline{\Omega}$ which is ε-orthogonal to $G(\Omega)$, the projection $\mathcal{P}_{H\nu}$ of $L^2(\Omega;\mathbb{R}^3)$ onto $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ is given by (11.6) with*

$$c_\ell \;=\; \int_{S_\ell} \varepsilon v \, d\sigma, \qquad 1 \;\le\; \ell \;\le\; L. \tag{11.8}$$

Proof. With v as in the theorem, (5.3) implies that $v \;=\; \varepsilon^{-1} \operatorname{curl} A + k$ for some $A \in V_{\tau 0}(\Omega)$ and $k \in \mathcal{H}_{\varepsilon\nu 0}(\Omega)$. Since v, ε^{-1}, k are continuous on $\overline{\Omega}$, so is $\operatorname{curl} A$ and then the surface integral of $\operatorname{curl} A$ over S_ℓ is zero from Stokes' theorem and a density argument. Hence (11.6) and (11.5) implies (11.8). □

For a general continuous vector field on $\overline{\Omega}$, (11.8) becomes

$$c_\ell \;=\; \int_{S_\ell} \varepsilon(v - \nabla\varphi) \, d\sigma,$$

where φ is the scalar potential of v defined as in Theorem 6.1. Let $W(\Omega)$ be the class of all irrotational L^2-vector fields on Ω. From the unweighted Hodge-Weyl decomposition and (i) of proposition 4.2, it follows that

$$W(\Omega) \;=\; G(\Omega) \oplus \mathcal{H}_{\nu 0}(\Omega). \tag{11.9}$$

This holds as the non-zero fields in $\mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$ have non-zero curls from the description of the basis of this space in [3, Section 10]. This leads to the following result—well-known when w is smooth—that is needed below.

Theorem 11.3. *Assume Ω satisfies (B1) and $w \in L^2(\Omega;\mathbb{R}^3)$ is a continuous irrotational vector field on Ω. Then there is a $\varphi \in H^1(\Omega)$ with $w \;=\; \nabla\varphi$ if and only if*

$$\int_\gamma w \;=\; 0 \quad \textit{for every closed } C^1\textit{-curve } \gamma \subset \Omega. \tag{11.10}$$

Proof. True if $\mathcal{H}_{\nu 0}(\Omega) \;=\; \{0\}$. When Ω is not simply connected, then the representation (11.9) implies that $w \;=\; \nabla\varphi + k$ for some $k \in \mathcal{H}_{\nu 0}(\Omega)$. Using the description of this space in [12] or [9, Chapter 9], this $k = 0$ if and only if (11.10) holds. □

12. The Prescribed Flux *div-curl* Problem

We now describe the L^2-solvability and well-posedness of various *div-curl* problems on domains which satisfy (B1). First consider the problem of finding finite-energy solutions subject to given boundary fluxes. Namely given a matrix valued function ε, the vector-valued function ω and scalar functions ρ and μ find solutions $v \in L^2(\Omega;\mathbb{R}^3)$ of the system

$$\operatorname{div}\big(\varepsilon(x)v(x)\big) = \rho(x) \quad \text{and} \tag{12.1}$$

$$\operatorname{curl} v(x) = \omega(x) \quad \text{on} \quad \Omega, \text{ with} \tag{12.2}$$

$$\big(\varepsilon(x)v(x)\big)\cdot\nu(x) = \mu(x) \quad \text{on } \partial\Omega. \tag{12.3}$$

Various results on the finite-energy solvability of this problem have been described by Picard in [16] and Saranen in [17] and [18]. They used operator-theoretic methods on Hilbert spaces. There is also a considerable literature on the classical solvability of these problems. Here we use variational principles and variational methods to obtain somewhat different, and sharper, results. Some results of this type have already been described in [4] for the case where $\varepsilon(x) \equiv e(x)I_3$ is everywhere a multiple of the identity matrix.

Throughout the following analysis we assume the following integrability conditions which are necessary for the functionals in the variational principles to be continuous.

Condition CF. $\rho \in L^{6/5}(\Omega)$, $\omega \in L^{6/5}(\Omega;\mathbb{R}^3)$ *and* $\mu \in L^{4/3}(\partial\Omega)$.

First note that if there is a solution of this system then the data must satisfy certain compatibility conditions

Proposition 12.1. *Assume (B1), (CF) and (E1) hold and* $v \in L^2(\Omega;\mathbb{R}^3)$ *is a weak solution of (12.1)–(12.3), then the data must satisfy*

$$\textbf{(NC1):}\qquad \int_\Omega \rho\, d^3x = \int_{\partial\Omega} \mu\, d\sigma,$$

$$\textbf{(NC2):}\qquad \int_\Omega \omega\cdot A\, d^3x = 0 \quad \textit{for all irrotational fields } A \in H^1_{\tau 0}(\Omega;\mathbb{R}^3).$$

Proof. Take $\varphi \equiv 1, u = \varepsilon v$ and substitute in (3.17). Then (NC1) follows. Multiply (12.2) by $\nabla\varphi$ with $\varphi \in C_c^\infty(\Omega)$ and use (3.16). Then $\langle\omega,\nabla\varphi\rangle = 0$, so ω is orthogonal to $G_0(\Omega)$. Similarly multiply (12.2) by $h \in \mathcal{H}_{\tau 0}(\Omega)$ and use (3.16). The class of irrotational fields in $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ is dense in $G_0(\Omega)\oplus\mathcal{H}_{\tau 0}(\Omega)$ from the analysis of Section 7 so (NC2) follows. □

These are necessary conditions for the system (12.1)–(12.3) to have a finite-energy (i.e., L^2) solution. Condition (NC2) is more recognizable in a differential form. The condition that ω is orthogonal to $G_0(\Omega)$ is the weak form of $\operatorname{div}\omega = 0$ on Ω. When $J \geq 1$, this condition also requires that ω be orthogonal to any field in $\mathcal{H}_{\tau 0}(\Omega)$. The description of these fields in Section 10 shows that this is equivalent to the condition that there be no net flux of ω through any boundary component Σ_j.

We next show that when these conditions hold, there is at least one finite-energy solution by using the decomposition (5.3). This particular representation leads to a decoupled system of elliptic boundary-value problems for the potentials φ and A. We then describe, and prove results about, variational principles for these potentials. Thus we seek $\varphi \in H^1(\Omega)$, $A \in V^1_{\tau 0}(\Omega)$ and $k \in \mathcal{H}_{\varepsilon\nu 0}(\Omega)$ such that

$$v(x) \;=\; \nabla\varphi(x) + \varepsilon(x)^{-1}\operatorname{curl} A(x) + k(x) \tag{12.4}$$

is a solution of (12.1)–(12.3). Substituting this in (12.1), φ must satisfy

$$\operatorname{div}\big(\varepsilon(x)\nabla\varphi(x)\big) \;=\; \rho(x) \quad \text{on } \Omega, \text{ and} \tag{12.5}$$

$$(\varepsilon\nabla\varphi)\cdot\nu \;=\; \mu \quad \text{on } \partial\Omega. \tag{12.6}$$

A weak solution of this system is a function φ in $H^1(\Omega)$ satisfying

$$\int_\Omega [(\varepsilon\nabla\varphi)\cdot\nabla\mathcal{X} + \rho\mathcal{X}]\, d^3x - \int_{\partial\Omega} \mu\mathcal{X}\, d\sigma \;=\; 0 \tag{12.7}$$

for all $\mathcal{X} \in H^1(\Omega)$. Consider the variational problem of minimizing the functional $\mathcal{D}$: $H^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}(\varphi) \;:=\; \int_\Omega \left[\frac{1}{2}(\varepsilon(\nabla\varphi)\cdot\nabla\varphi) + \rho\varphi\right]\, d^3x - \int_\Omega \mu\varphi\, d\sigma. \tag{12.8}$$

The following theorem provides an existence result for this variational principle and thus also for H^1-solutions of (12.7).

Theorem 12.2. *Assume (B1), (CF) and (E1) hold. Then $\mathcal{D}$ defined by (12.8) is bounded below on $H^1(\Omega)$ if and only if (NC1) holds. In this case, there is a unique function $\hat{\varphi}$ which minimizes $\mathcal{D}$ on $H^1_m(\Omega)$. A function $\tilde{\varphi}$ minimizes $\mathcal{D}$ on $H^1(\Omega)$ if and only if it satisfies (12.7). In this case $\varphi = \tilde{\varphi} + c$ for some constant c.*

Proof. When $\varphi(x) \equiv c$ is constant on Ω,

$$\mathcal{D}(c) = c\int_\Omega \rho\, d^3x - c\int_{\partial\Omega} \mu\, d\sigma.$$

Letting $|c| \to \infty$, it follows that $\mathcal{D}$ is unbounded below on $H^1(\Omega)$ if and only if (NC1) holds. Let $\mathcal{P}_m$ be the projection of $H^1(\Omega)$ onto $H^1_m(\Omega)$ defined in Section 5, $\mathcal{D}_u(\mathcal{P}_m\varphi) = \mathcal{D}_u(\varphi)$ for all $\varphi \in H^1(\Omega)$. So a function $\tilde{\varphi}$ minimizes $\mathcal{D}$ on $H^1(\Omega)$ if and only if $\mathcal{P}_m\tilde{\varphi}$ minimizes $\mathcal{D}$ on $H^1_m(\Omega)$. When $\varphi \in H^1(\Omega)$, then $\varphi \in L^q(\Omega)$ for $q \le 6$ and its trace on $\partial\Omega \in L^r(\partial\Omega)$ for $r \le 4/3$ from the usual imbedding theorems. Thus the last two terms in (12.8) are continuous linear functionals on $H^1(\Omega)$ under our assumptions on ρ and μ, and $\mathcal{D}$ is a continuous, convex functional on $H^1(\Omega)$. Hence it is weakly lower semi-continuous. From Poincaré's inequality as in (6.4), there are positive constants c_0, c_1, c_2 such that

$$\mathcal{D}(\varphi) \ge c_0\|\nabla\varphi\|^2 - c_1\|\rho\|_{6/5}\|\nabla\varphi\| - c_2\|\mu\|_{\partial\Omega,4/3}\|\nabla\varphi\| \tag{12.9}$$

for $\varphi \in H^1_m(\Omega)$. Hence $\mathcal{D}$ is strictly convex and coercive on $H^1_m(\Omega)$ and attains a unique minimizer. $\mathcal{D}$ is Gateaux differentiable on $H^1(\Omega)$ with

$$\langle \mathcal{D}'(\varphi), \mathcal{X}\rangle = \int_\Omega [(\varepsilon\nabla\varphi)\cdot\nabla\mathcal{X} + \rho\mathcal{X}]\; d^3x - \int_{\partial\Omega} \mu\mathcal{X}\, d\sigma. \tag{12.10}$$

At a minimizer of $\tilde{\varphi}$ of $\mathcal{D}$ on $H^1(\Omega)$, (12.10) is zero for all $\mathcal{X} \in H^1(\Omega)$, so $\tilde{\varphi}$ satisfies (12.7). Conversely if $\tilde{\varphi}$ satisfies (12.7) then $\tilde{\varphi}$ is a critical point of $\mathcal{D}$ on $H^1(\Omega)$. The convexity of $\mathcal{D}$ implies that such solutions minimize $\mathcal{D}$ and the theorem follows. □

Corollary 12.3. *Assume the conditions of the theorem and (NC1). If $\hat{\varphi}$ minimizes $\mathcal{D}$ on $H^1_m(\Omega)$ then there are constants k_1, k_2 which depend only on Ω, ε such that*

$$\|\nabla \hat{\varphi}\|_2 \;\leq\; k_1(\|\rho\|_{6/5} + k_2\|\mu\|_{4/3,\partial\Omega}. \tag{12.11}$$

Proof. Since $0 \in H^1_m(\Omega)$, the value of this variational problem is less than or equal to 0. The inequality (12.9) implies that $\hat{\varphi}$ satisfies

$$c_0 \, \|\nabla \varphi\| \;\leq\; c_1 \, \|\rho\|_{6/5} + c_2 \, \|\mu\|_{\partial\Omega, 4/3} \, ,$$

where c_0 depends on ε, c_1, c_2 depend only on Ω. Thus (12.11) follows. □

Similarly if we take curls of both sides of (12.4), the vector potential satisfies

$$\operatorname{curl}\big(\varepsilon(x)^{-1} \operatorname{curl} A(x)\big) \;=\; \omega(x) \quad \text{on } \Omega. \tag{12.12}$$

The weak form of this is to find $A \in H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ which satisfies

$$\int_\Omega [(\varepsilon^{-1} \operatorname{curl} A) \cdot \operatorname{curl} B - \omega \cdot B] \; d^3x \;=\; 0 \tag{12.13}$$

for all $B \in H^1_{\tau 0}(\Omega; \mathbb{R}^3)$. Consider the variational problem of minimizing $\mathcal{C}_0 : H^1_{\tau 0}(\Omega; \mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{C}_0(A) \;:=\; \int_\Omega \left[\frac{1}{2}(\varepsilon^{-1} \operatorname{curl} A) \cdot \operatorname{curl} A - \omega \cdot A \right] \; d^3x. \tag{12.14}$$

This functional is not coercive on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$. In fact, $\mathcal{C}_0$ is bounded below on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ if and only if (NC2) holds. Then the following existence theorem holds.

Theorem 12.4. *Assume (B1), (CF) and (E1) hold. Then $\mathcal{C}_0$ is bounded below on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ if and only if (NC2) holds. When (NC2) holds there are minimizers of $\mathcal{C}_0$ on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ and a unique minimizer $\hat{A}$ of $\mathcal{C}_0$ on $Z_{\tau 0}(\Omega)$. A field $\tilde{A}$ minimizes $\mathcal{C}_0$ on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ if and only if it satisfies (12.13). In this case*

$$\tilde{A} \;=\; \hat{A} + \nabla \varphi + h \quad \text{ for some } \varphi \in H^1_0(\Omega) \text{ and } h \in \mathcal{H}_{\tau 0}(\Omega). \tag{12.15}$$

Proof. The functional $\mathcal{C}_0$ is similar to the functional $\mathcal{C}_u$ treated in Theorem 8.3; the differences are a factor of 2 and the linear terms. Let P_τ be the projection of $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ onto $Z_{\tau 0}(\Omega)$, then

$$\mathcal{P}_\tau A \;:=\; A - \nabla \varphi - h,$$

where $\varphi \in H^1_0(\Omega)$ and $h := \mathcal{P}_{H2} A \in \mathcal{H}_{\tau 0}(\Omega)$ are as in equation (7.5). From (NC2), we have that

$$\int_\Omega \omega \cdot (\nabla \varphi + h) \; d^3x \;=\; 0$$

for all such φ, h. Thus $\mathcal{C}_0(\mathcal{P}_\tau A) = \mathcal{C}_0(A)$ for all $A \in H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ and $\mathcal{C}_0$ is bounded below on $H^1_{\tau 0}(\Omega; \mathbb{R}^3)$ as it is bounded below on $Z_{\tau 0}(\Omega)$. The other parts of this theorem are now proved just as in Section 8, □

Corollary 12.5. *Assume the conditions of the theorem and (NC2). If $\hat{A}$ minimizes $\mathcal{C}_0$ on $Z_{\tau 0}(\Omega)$, then there is a constant k_3 which depend only on Ω, ε such that*

$$\left\|\operatorname{curl} \hat{A}\right\|_2 \ \leq \ k_3 \ \|\omega\|_{6/5}. \tag{12.16}$$

Proof. Since $0 \in Z_{\tau 0}(\Omega)$, the value of this variational problem is less than or equal to 0. The functional $\mathcal{C}_0$ is coercive on $Z_{\tau 0}(\Omega)$ with

$$\mathcal{C}_0(A) \ \geq \ \frac{1}{2e_1} \left\|\operatorname{curl} A\right\|^2 - c_3 \|\omega\|_{6/5} \left\|\operatorname{curl} A\right\| \tag{12.17}$$

using the fact that the imbedding of $H^1(\Omega;\mathbb{R}^3)$ into $L^6(\Omega;\mathbb{R}^3)$ is continuous and Holder's inequality. Here c_3 depends only on Ω. Thus the minimizer of $\mathcal{C}_0$ on $Z_{\tau 0}(\Omega)$ satisfies (12.16). ☐

When $\tilde{\varphi}$ is a minimizer of $\mathcal{D}$ and $\tilde{A}$ is a minimizer of $\mathcal{C}_0$ as above then, by linearity,

$$\hat{v}(x) \ = \ \nabla\tilde{\varphi}(x) + \varepsilon(x)^{-1} \operatorname{curl} \tilde{A}(x) + k(x) \tag{12.18}$$

a solution of (12.1)–(12.3) for each $k \in \mathcal{H}_{\varepsilon\nu 0}(\Omega)$. That is, there is an L-parameter family of solutions of this prescribed flux *div-curl* system. To specify a unique solution we must further impose conditions that select a unique ε-harmonic field k. When Ω is simply connected no further conditions are required. If however, $\beta_1(\Omega) = L \geq 1$, then (11.6) and (11.7) show that the solution is determined provided we know the values of the inner products $\langle v, k^{(\ell)}\rangle_{\varepsilon}$. These are always well-defined and from Theorem 11.2, when the solution v is smooth enough, they are the familiar flux integrals through the cutting surfaces;

$$\int_{S_\ell} \varepsilon v \ = \ \kappa_l, \qquad 1 \ \leq \ \ell \ \leq \ L. \tag{12.19}$$

Note that this just involves solving a linear equation for the coefficients c_l in the representation (11.6) of k in terms of the basis of $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$.

This discussion leads to the following well-posedness result for this *div-curl* system.

Theorem 12.6. *Assume (B1), (CF), (NC1), (NC2) and (E1) hold. If Ω is simply connected then there is a unique solution $\hat{v} \in L^2(\Omega;\mathbb{R}^3)$ of (12.1)–(12.3). When $\beta_1(\Omega) = L \geq 1$, there is an L-parameter family of solutions of this system. If $S_1, \ldots, S_L$ are L independent cutting surfaces for the region Ω, and $\kappa := (\kappa_1, \ldots, \kappa_L)$ is given, there is a unique solution of this system which also satisfies (12.19).*

The corollaries to the above existence results provide inequalities for the minima of these variational problems which yield the following estimate for these solutions.

Theorem 12.7. *Assume the conditions of Theorem 12.6 hold. Then there are constants C_j depending only on Ω, ε such that the unique solution of (12.1)–(12.3) and (12.19) satisfies*

$$\|v\|_2 \ \leq \ C_1(\|\rho\|_{6/5} + \|\omega\|_{6/5}) + C_2\|\mu\|_{4/3,\partial\Omega} + C_3 \left|\tilde{\kappa}\right|, \tag{12.20}$$

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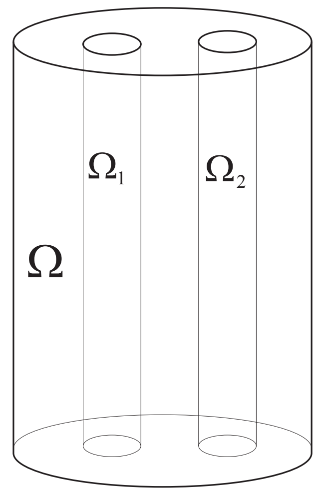

FIGURE 1. Ω is the region exterior to the tubes Ω_1, Ω_2 and with connected boundary $\partial\Omega$. This region has $J = 0, L = 2$. To specify a unique solution of the given flux *div-curl* problem in this region, the flux of v through two independent cutting surfaces of the region must be prescribed.

where $\tilde{\kappa} := (\kappa_1, \ldots, \kappa_L)$.

Proof. The unique solution is given by (12.4) and, from (5.3), this is an ε-orthogonal decomposition. Hence

$$\|v\|_\varepsilon^2 \;=\; \|\nabla\hat{\varphi}\|_\varepsilon^2 + \left\|\operatorname{curl}\hat{A}\right\|_\varepsilon^2 + \|k\|_\varepsilon^2 .$$

The norm $\|k\|$ depends linearly on $|\tilde{\kappa}|$ as it is the solution of a nonsingular finite dimensional linear equation. Use the inequalities (12.11) and (12.16) and the fact that the ε norm on L^2 is equivalent to the usual L^2-norm then (12.20) follows. $\square$

13. The Tangential *div-curl* Problem

Similar analyses may be used to describe the well-posedness of the *div-curl* problem of solving (12.1)–(12.2) subject to given tangential boundary data

$$v(x) \wedge \nu(x) \;=\; \eta(x) \text{ on } \partial\Omega. \tag{13.1}$$

This is sometimes called the *electric-type* boundary-value problem for the *div-curl* system. Results on this problem have been described previously by Saranen [17] and [18], and by Picard [16]. They used Hilbert space methods to obtain certain existence theorems.

We require the following integrability conditions on the data to ensure that our functionals are continuous.

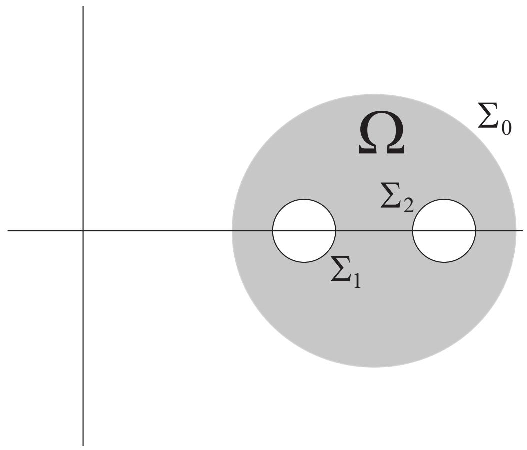

FIGURE 2. Ω is the region obtained by rotating the cross-section depicted above about the z-axis. It is a torus with 2 interior tori excised. The boundary $\partial\Omega$ has 3 components; the exterior surface Σ_0 and 2 interior surfaces Σ_1, Σ_2. This region has $J = 2, L = 3$. To specify a unique solution of the given flux *div-curl* problem in this region, the flux of v through three independent cutting surfaces of the region must be prescribed.

Condition CT. $\rho \in L^{6/5}(\Omega)$, $\omega \in L^{6/5}(\Omega;\mathbb{R}^3)$, *and* $\eta \in L^{4/3}(\partial\Omega;\mathbb{R}^3)$.

When there is an L^2-solution of this system, (3.18) leads to the following.

Proposition 13.1. *Assume (B1) and (CT) hold and* $v \in L^2(\Omega;\mathbb{R}^3)$ *is an weak solution of (12.1), (12.2) and (13.1) then the data must satisfy*

(NC3): η *is tangential on* $\partial\Omega$, *and*

(NC4): $\int_\Omega \omega\cdot A \, d^3x + \int_{\partial\Omega} \eta\cdot A \, d\sigma \;=\; 0$ *for every irrotational field* $A \in H^1(\Omega;\mathbb{R}^3)$.

Proof. Take scalar products of (13.1) with $\nu(x)$, then (NC3) holds. Substitute (12.2) and (13.1) in (3.18) then (NC4) holds for all $A \in H^1(\Omega;\mathbb{R}^3)$. □

The necessary condition (NC4) is stronger than condition (NC2) in the previous section. It may be regarded as a weak form of certain equations on both Ω and $\partial\Omega$. From the analysis in Section 7, the class of irrotational fields in $H^1(\Omega;\mathbb{R}^3)$ is precisely $G^1(\Omega) \oplus \mathcal{H}_{\nu 0}(\Omega)$, where $G^1(\Omega) := \{\nabla\varphi : \varphi \in H^2(\Omega)\}$. When ω, η are smooth, upon substituting $\nabla\varphi$ for A in (NC4), it follows that ω must be solenoidal on Ω and also that

$$\operatorname{div}_\partial \eta + \omega\cdot\nu \;=\; 0 \quad \text{on } \partial\Omega. \tag{13.2}$$

Here $\operatorname{div}_\partial$ is the surface divergence on the manifold(s) constituting $\partial\Omega$. This compatibility condition is a continuity equation which may be regarded as a continuous version of Kirchoff's rule for currents.

When the region Ω is not simply connected, (NC4) also implies L conditions of the form

$$\int_\Omega \omega \cdot \tilde{k}^{(\ell)} \, d^3x + \int_{\partial\Omega} \eta \cdot \tilde{k}^{(\ell)} \, d\sigma \;=\; 0, \qquad 1 \le \ell \le L. \tag{13.3}$$

Here $\{\tilde{k}^{(1)}, \ldots, \tilde{k}^{(L)}\}$ is a basis of $\mathcal{H}_{\nu 0}(\Omega)$ as in Section 11.

We next show that if the data obeys (CT) and (NC4), then there is at least one finite-energy solution of this boundary-value problem. This is done by seeking solutions of the form (5.4). Assume

$$v(x) \;=\; \nabla\psi(x) + \varepsilon(x)^{-1} \operatorname{curl} B(x) + h(x), \tag{13.4}$$

with $\psi \in H_0^1(\Omega), B \in V_{\nu 0}^1(\Omega)$ and $h \in \mathcal{H}_{\varepsilon\tau 0}(\Omega)$ is a solution of (12.1), (12.2) and (13.1). Substitute this in (12.1), then $\psi \in H_0^1(\Omega)$ satisfies

$$\operatorname{div}\big(\varepsilon(x)\nabla\psi(x)\big) \;=\; \rho(x) \quad \text{ on } \Omega. \tag{13.5}$$

A weak solution of this system is a function ψ in $H_0^1(\Omega)$ which satisfies

$$\int_\Omega [(\varepsilon\nabla\psi)\cdot\nabla\mathcal{X} + \rho\mathcal{X}\,] \, d^3x \;=\; 0 \qquad \text{for all } \mathcal{X} \in H_0^1(\Omega). \tag{13.6}$$

Consider the variational problem of minimizing $\mathcal{D}_1 : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}_1(\psi) \;:=\; \int_\Omega \left[\frac{1}{2}(\varepsilon(\nabla\psi)\cdot\nabla\psi) + \rho\psi\right] d^3x. \tag{13.7}$$

Theorem 13.2. *Assume (B1), (CF) and (E1) hold. Then $\mathcal{D}_1$ defined by (13.7) has a unique minimizer $\hat{\psi}$ on $H_0^1(\Omega)$ and $\hat{\psi}$ is the unique solution of (13.6) in $H_0^1(\Omega)$.*

The proof of this is similar to, but simpler than that of Theorem 12.2.

Take curls of both sides of (13.4), then the vector potential satisfies

$$\operatorname{curl}\big(\varepsilon(x)^{-1}\operatorname{curl} B(x)\big) \;=\; \omega(x) \quad \text{and} \tag{13.8}$$

$$\operatorname{div} B \;=\; 0 \qquad \text{on } \Omega. \tag{13.9}$$

$$B\cdot\nu \;=\; 0 \quad \text{and} \quad (\varepsilon^{-1}\operatorname{curl} B)\wedge\nu \;=\; \eta \qquad \text{on } \partial\Omega. \tag{13.10}$$

The weak form of this system is to find $B \in H_{\nu 0}^1(\Omega;\mathbb{R}^3)$ which satisfies

$$\int_\Omega [(\varepsilon^{-1}\operatorname{curl} B)\cdot\operatorname{curl} C - \omega\cdot C] \, d^3x - \int_{\partial\Omega} \eta\cdot C \, d\sigma \;=\; 0 \quad \text{for all } C \in H_{\nu 0}^1(\Omega;\mathbb{R}^3). \tag{13.11}$$

Consider the variational problem of minimizing $\mathcal{C} : H^1(\Omega;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{C}(B) \;:=\; \mathcal{C}_0(B) - \int_{\partial\Omega} \eta\cdot B \, d\sigma, \tag{13.12}$$

with $\mathcal{C}_0$ being the functional from (12.14).

Theorem 13.3. *Assume (B1), (CT) and (E1) hold. Then* $\mathcal{C}$ *is bounded below on* $H^1(\Omega;\mathbb{R}^3)$ *if and only if (NC4) holds. In this case there is a unique minimizer* $\hat{B}$ *of* $\mathcal{C}$ *on* $Z_{\nu 0}(\Omega)$*. A field* $\tilde{B}$ *minimizes* $\mathcal{C}$ *on* $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ *if and only if it satisfies (13.11). In this case*

$$\tilde{B} \;=\; \hat{B} + \nabla\varphi + k \quad \text{for some } \varphi \in H^1(\Omega) \text{ and } k \in \mathcal{H}_{\nu 0}(\Omega). \tag{13.13}$$

Proof. This proof is similar to that of Theorem 12.4 except now we use the projection $\mathcal{P}_\nu$ of $H^1_{\nu 0}(\Omega;\mathbb{R}^3)$ onto $Z_{\nu 0}(\Omega)$ defined as in equation (7.3). Then $\mathcal{C}(\mathcal{P}_\nu A) = \mathcal{C}(A)$ for all $A \in H^1(\Omega;\mathbb{R}^3)$ if and only if (NC4) holds. In this case, $\mathcal{C}$ is bounded below on $H^1(\Omega;\mathbb{R}^3)$ as it is bounded below on $Z_{\nu 0}(\Omega)$, if (NC4) fails then the functional is unbounded below and the rest of the proof follows as before. □

By linearity, when $\tilde{\psi}$ is a minimizer of $\mathcal{D}_1$ and $\tilde{B}$ is a minimizer of $\mathcal{C}$ as above then

$$\hat{v}(x) \;=\; \nabla\tilde{\psi}(x) + \varepsilon(x)^{-1}\operatorname{curl}\tilde{B}(x) + h(x) \tag{13.14}$$

is a solution of (12.1)–(12.2) and (13.1) for each $h \in \mathcal{H}_{\varepsilon\tau 0}(\Omega)$. If the region Ω has no holes, then $h = 0$. When $J \geq 1$ there is a J-parameter family of solutions of this tangential *div-curl* problem. To specify a unique solution, additional conditions must be imposed that determine the ε-harmonic component h. From (10.7) and (10.8), this is done when the coefficients $\langle v, h^{(k)}\rangle_\varepsilon$ are given. From Theorem 10.2, provided the solution is continuous, this amounts to specifying the line integrals, or potential differences,

$$\int_{\xi_j} v \;=\; \alpha_j, \qquad 1 \;\leq\; j \;\leq\; J, \tag{13.15}$$

where each ξ_j is a C^1 curve in $\overline{\Omega}$ which joins Σ_0 to another component Σ_j of $\partial\Omega$.

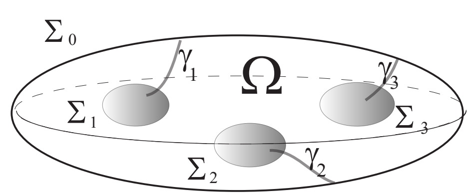

FIGURE 3. Ω is the region interior to the closed surface Σ_0 and exterior to the three cavities with surfaces $\Sigma_1, \Sigma_2, \Sigma_3$. This region has $J = 3, L = 0$. The tangential *div-curl* problem in this region is well-posed when three extra conditions are prescribed. If the solution is continuous, these may be the line integrals of v along paths such as $\gamma_1, \gamma_2, \gamma_3$.

Thus we have the following well-posedness result for the tangential *div-curl* boundary-value problem.

Theorem 13.4. *Assume (B1), (CT), (NC4) and (E1) hold. If $\partial\Omega$ has only one component, there is a unique solution $\hat{v} \in L^2(\Omega;\mathbb{R}^3)$ of (12.1)–(12.2) and (13.1). When $\beta_2(\Omega) = J \geq 1$, there is an J-parameter family of solutions of this system. For given $\alpha := (\alpha_1, \ldots, \alpha_J)$, there is a unique solution of this system which also satisfies (13.15).*

Using the same methods as in the previous section, one may obtain norm inequalities for the minima of each of these variational problems. Use the ε-orthogonality of the decomposition (13.4), to obtain the following estimate about the solutions of this problem.

Theorem 13.5. *Assume the conditions of Theorem 13.4 hold. Then there are constants C_j depending only on Ω, ε such that the unique solution of (12.1)–(12.2) and (13.1), with (13.15) satisfies*

$$\|u\|_2 \;\leq\; C_1(\|\rho\|_{6/5} + \|\omega\|_{6/5}) + C_2\|\eta\|_{4/3,\partial\Omega} + C_4\,|\tilde{\alpha}|\,, \tag{13.16}$$

where $\tilde{\alpha} := (\alpha_1, \ldots, \alpha_J)$.

14. Appendix

The following is a list of notation of spaces used in this paper for the normal and tangential *div-curl* boundary-value problems. The section where they are defined is also given.

- $C(\overline{\Omega}), C(\overline{\Omega};\mathbb{R}^3)$ and $C(\partial\Omega)$ are spaces of continuous functions or fields defined on $\overline{\Omega}$ and in section 3.
- $C_c^\infty(\Omega), C_c^\infty(\Omega;\mathbb{R}^3)$ are spaces of smooth functions and fields defined in Section 3.
- $\mathrm{Curl}_\varepsilon(\Omega), \mathrm{Curl}_{\varepsilon\nu 0}(\Omega), \mathrm{Curl}_{\varepsilon\tau 0}(\Omega)$ are spaces of weighted curl fields defined in Section 4.
- $G(\Omega), G_0(\Omega)$ are spaces of gradient fields defined in Section 4. $G^1(\Omega)$ is defined and used in Section 12.
- $H^1(\Omega), H_m^1(\Omega), H_0^1(\Omega)$ are standard Sobolev-Hilbert spaces of functions introduced in Section 6. $H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega), H^2(\Omega)$ are also used.
- $H^1(\Omega;\mathbb{R}^3), H_{\nu 0}^1(\Omega;\mathbb{R}^3), H_{\tau 0}^1(\Omega;\mathbb{R}^3)$ are each Hilbert spaces of vector fields defined in Section 3.
- $H(\mathrm{curl},\Omega), H(\mathrm{div},\Omega)$ and $H_{DC}(\Omega)$ are subspaces of $L^2(\Omega;\mathbb{R}^3)$ defined in section 3.
- $\mathcal{H}_{\varepsilon\nu 0}(\Omega), \mathcal{H}_{\varepsilon\tau 0}(\Omega)$ are spaces of ε-harmonic fields defined in Section 5.
- $\mathcal{H}_{\nu 0}(\Omega), \mathcal{H}_{\tau 0}(\Omega)$ are spaces of harmonic fields defined in Sections 5 and 7.
- $L^p(\Omega), L_{\mathrm{loc}}^1(\Omega), L^1(\partial\Omega; d\sigma)$ are Lebesgue spaces of measurable functions defined in Section 3.
- $L^p(\Omega;\mathbb{R}^3), L^1(\partial\Omega;\mathbb{R}^3)$ are Lebesgue spaces of measurable vector fields defined in Section 3.
- $V_\varepsilon(\Omega), V_{\varepsilon\nu 0}(\Omega)$ are L^2 spaces of solenoidal vector fields defined in Section 4.
- $V_{\nu 0}^1(\Omega)$, $V_{\tau 0}^1(\Omega)$ are spaces of H^1 solenoidal fields defined in Section 7.
- $W^{1,p}(\Omega), W^{1,p}(\Omega;\mathbb{R}^3)$ are Sobolev spaces of functions and fields defined in Section 3.
- $Z_{\nu 0}(\Omega), Z_{\tau 0}(\Omega)$ are subspaces defined in Section 7.

There also are many projections defined in this paper. The following is a listing of them; $\mathcal{P}_V$ is the generic form of a projection onto a closed subspace V of a Hilbert space H. Some of the projections have domain $L^2(\Omega;\mathbb{R}^3)$, the domain is listed when it is not $L^2(\Omega;\mathbb{R}^3)$.

- $\mathcal{P}_m$ is the projection of $H^1(\Omega)$ onto $H^1_m(\Omega)$ defined in Section 6.
- $\mathcal{P}_G$ is the projection of $H^1(\Omega;\mathbb{R}^3)$ into itself defined in Section 7. $\mathcal{P}_S := I - \mathcal{P}_G$ is also defined there.
- $\mathcal{P}_{H1}$ and $\mathcal{P}_{H2}$ are projections of $H^1(\Omega;\mathbb{R}^3)$ onto the spaces of harmonic fields $\mathcal{H}_{\nu 0}(\Omega)$ and $\mathcal{H}_{\tau 0}(\Omega)$ defined in Section 7.
- $\mathcal{P}_\nu$ is the projection of $H^1(\Omega;\mathbb{R}^3)$ onto $Z_{\nu 0}(\Omega)$ defined in Section 7.
- $\mathcal{P}_{G0}$ and $\mathcal{P}_\tau$ are the projections of $H^1_{\tau 0}(\Omega;\mathbb{R}^3)$ onto $G_0(\Omega)$ and $Z_{\tau 0}(\Omega)$ defined in Section 7.
- $\mathcal{P}_{h\tau}$ is the projection onto $\mathcal{H}_{\varepsilon\tau 0}(\Omega)$ defined in Section 10.
- $\mathcal{P}_{h\nu}$ is the projection onto $\mathcal{H}_{\varepsilon\nu 0}(\Omega)$ defined in Section 11.

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