REPRODUCING KERNELS FOR HILBERT SPACES OF REAL HARMONIC FUNCTIONS.

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Abstract. This paper studies a family of Hilbert spaces of real harmonic functions on bounded regions in $\mathbb{R}^n$ and will show that, for a range of values of s, they are reproducing kernel Hilbert spaces. The spaces are characterized by their boundary traces and the inner products are defined via their expansions in the harmonic Steklov eigenfunctions of the region. The reproducing kernels will then be described explicitly in terms of the Steklov eigenfunctions. Expansions for some of the standard integral operators defining the solutions of Dirichlet, Robin and Neumann boundary value problems for Laplace's equation will also be derived and relationships of these operators to some reproducing kernels will be described

1. Introduction

This paper will introduce a family of Hilbert spaces $\mathcal{H}^s(\Omega),\ s \in \mathbb{R}$ of real harmonic functions on bounded regions Ω in $\mathbb{R}^n,\ n \geq 2$ and show that, for $s > 1/2$, these spaces are reproducing kernel Hilbert spaces. The least value of s for which the spaces are RKH spaces is an interesting open question. The reproducing kernels will be described explicitly in terms of the Steklov bases of the spaces.

These spaces are defined using the Steklov eigenfunctions of the Laplacian for the region. The different spaces are characterized by the regularity of their boundary traces in the sense of the spectral trace theory recently described in Auchmuty [4]. This theory holds on quite general bounded regions, including bounded Lipschitz regions. Natural (spectral) inner products for these spaces are introduced and isometric isomorphism between the spaces of harmonic functions on the region and trace spaces are described.

In [17], J-L Lions described a reproducing kernel for the space of all L^2- harmonic functions on a region with a very smooth boundary. His paper introduced a variational principle for the kernel and then used a control theoretic approach to obtain his results. The reproducing kernel he found involved the Green's function for a biharmonic operator. In one of his last papers [18], Lions generalized the L^2-results to a continuous family of spaces of harmonic functions. The family is characterized by the regularity of the boundary data and uses the boundary values to generate the inner product. To do this,

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Lions required that the boundary be a smooth Riemannian manifold and used properties of solutions of the Laplace-Beltrami operator on the boundary. Recently Engliš, Lukkassen, Peetre and Persson, [12] extended and improved Lion's results in a number of different directions. They provided a nice introduction to reproducing kernel Hilbert spaces, showed how the reproducing kernel depended on certain Green's functions and derived integral formulae for some special planar regions.

As may be seen by comparing the results of [18] or [12] with those given here, the use of Steklov eigenfunctions changes, and considerably simplifies, much of the analysis. Since the reproducing kernel of a Hilbert space depends on the inner product, the results here differ from the previous results. However, all the papers describe relationships between the Poisson kernel and the reproducing kernel.

Much of the analysis here depends on results about the harmonic Steklov eigenfunctions of the region. First they are used to define the trace spaces for a general class of bounded regions in $\mathbb{R}^n$. Then they are used to describe bases for the various spaces of harmonic functions that yield spectral representations of the Poisson kernel and the solution of Robin and Neumann boundary value problems for Laplace's equation on Ω. Finally the reproducing kernels have simple expressions as Steklov expansions. The results used depend on the characterization, and completeness, of Steklov eigenfunctions described earlier in Auchmuty [3] and [4].

Many of the results here are related to topics studied in Part B of the monograph of Bergman and Schiffer [6]. There Bergman and Schiffer described many results about special integral operators associated with different boundary value problems for perturbations of the Laplacian on planar regions. They commented that their results did not apply to the Laplacian itself, but sought various reproducing properties for the systems that they studied. See, for example, Part B, chapter 1, section 6. Here we shall show that *kernel function* of Laplacian for the domain Ω can be related to the projection onto the space of harmonic functions on Ω as described in section 9. Also other results for Robin and Neumann boundary value problems for Laplace's equation that are similar to their results for simple Scroedinger - type operators will be proved. Again the use of Steklov expansions simplifies many of the results and provides sharp regularity results involving the spaces $H^s(\partial\Omega)$ and $\mathcal{H}^s(\Omega)$.

2. Definitions and Notation.

Throughout this paper we shall work on a bounded region Ω of $\mathbb{R}^n$. A region is a non-empty, connected, open subset of $\mathbb{R}^n$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. The definitions and terminology of Evans and Gariepy [13], will be followed except that σ, $d\sigma$, respectively, will represent Hausdorff $(n-1)-$dimensional measure and integration with respect to this measure. All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and derivatives should be taken in a weak sense.

The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$, $1 \leq p \leq \infty$ are defined in the standard manner and have the usual p-norms denoted by $\|u\|_p$ and $\|u\|_{p,\partial\Omega}$. When $p = 2$, these spaces will be Hilbert spaces with inner products

$$\langle u, v \rangle := \int_{\Omega} u(x)\, v(x)\, dx \qquad \text{and} \quad \langle u, v \rangle_{\partial\Omega} := |\partial\Omega|^{-1} \int_{\partial\Omega} u\, v\ d\sigma.$$

To simplify many formulae, we shall use the normalized surface area measure defined by $\tilde{\sigma}(E) := |\partial\Omega|^{-1}\sigma(E)$ for Borel subsets E of the boundary $\partial\Omega$. Let $H^1(\Omega)$ be the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1- inner product

$$[u, v]_1 := \int_{\Omega} [u(x).v(x) \ + \ \nabla u(x) \cdot \nabla v(x)]\ dx. \tag{2.1}$$

Here ∇u is the gradient of the function u and the associated norm is denoted $\|u\|_{1,2}$.

The essential assumption about Ω that will be used throughout this paper is the following.

(B): *Ω is a bounded region with a boundary $\partial\Omega$ for which the Gauss-Green, Rellich and compact trace theorems hold.*

There are a variety of conditions on the regularity of the boundary that guarantee this condition. Condition (B) holds when the boundary is the disjoint union of a finite number of closed C^1-manifolds each with finite surface area. There is an extensive, unfortunately very scattered, literature on the validity of each of these results when the boundary is less smooth. Condition (B) may hold when the boundary consists of locally Lipschitz surfaces, or possibly even has cuspidal singularities of special types,

Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(n) := 2n/(n-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $n = 2$.

There are a number of different criteria on Ω and $\partial\Omega$ that imply this result. One version is theorem 1 in section 4.6 of [13]. See also Amick [1]. DiBenedetto [11], theorem 14.1 of chapter 9 shows that the result holds when Ω is bounded and satisfies a "cone property". Adams and Fournier [2], chapter 6 provide a thorough treatment of conditions for this result in and show that it holds even for some classes of unbounded regions.

When u is a Lipschitz continuous function on $\overline{\Omega}$, the trace map Γu is the restriction of u to $\partial\Omega$. Under a variety of conditions on the boundary, this trace map may be extended to be a continuous map from $H^1(\Omega)$ to $L^2(\partial\Omega,\, d\sigma)$. The region Ω is said to satisfy a *compact trace theorem* provided the trace mapping $\Gamma : H^1(\Omega) \to L^2(\partial\Omega,\, d\sigma)$ is compact.

Evans and Gariepy [13], section 4.3 show that Γ is continuous when the boundary is Lipschitz. Similarly Theorem 1.5.1.10 of Grisvard [15] proves an inequality that implies the compact trace theorem under conditions on the boundary. This inequality is also proved in [11], chapter 9, section 18 under stronger regularity conditions on the boundary. Some discussion of general regions for which (B) holds may be found in section 3 of [7].

Often u will be used in place of Γu for the trace of a function on $\partial\Omega$. Suppose there is an outward unit normal $\nu(y)$ at σ *a.e.* point y of $\partial\Omega$. The *Gauss-Green* theorem for Ω

is the statement that

$$\int_\Omega u(x) D_j\, v(x)\ dx \ = \ |\partial\Omega| \int_{\partial\Omega} u\, v\, \nu_j\ d\tilde{\sigma} \ - \ \int_\Omega v(x) D_j\, u(x)\ dx \quad \text{for } 1 \le j \le n. \tag{2.2}$$

and all $u,\ v$ in $H^1(\Omega)$. This result is usually proved as a by-product of a trace theorem.

The following equivalent inner product on $H^1(\Omega)$ will be used extensively.

$$[u, v]_\partial := \int_\Omega \nabla u \cdot \nabla v\ dx + \int_{\partial\Omega} u\ v\ \ d\tilde{\sigma}. \tag{2.3}$$

The corresponding norm will be denoted by $\|u\|_\partial$. The proof that this norm is equivalent to the usual $(1, 2)-$norm on $H^1(\Omega)$ when (B) holds is Corollary 6.2 of [3] and also is part of theorem 21A of [21].

3. The Space $\mathcal{H}(\Omega)$

Henceforth we shall assume Ω is a region in $\mathbb{R}^n$ which satisfies (B). A function $u \in H^1(\Omega)$ is said to be *harmonic on* Ω provided it is a solution of Laplace's equation in the usual weak sense. Namely

$$\int_\Omega \nabla u \cdot \nabla \varphi\ dx \ = \ 0 \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{3.1}$$

Here $C_c^1(\Omega)$ is the set of all C^1-functions on Ω with compact support in Ω. From Weyl's lemma see [9], Chapter 2, section 3, proposition 9, any such solution is C^∞ on Ω - but the solution need not be continuous on the closure $\overline{\Omega}$.

Define $\mathcal{H}(\Omega)$ to be the subspace of all harmonic functions in $H^1(\Omega)$. Sometimes this will be called the space of all *weakly harmonic functions* on Ω, or the space of all *finite energy* harmonic functions on Ω.

When (B) holds, the completion of $C_c^1(\Omega)$ in the H^1-norm is the usual Sobolev space $H_0^1(\Omega)$. Then (3.1) is equivalent to saying that $\mathcal{H}(\Omega)$ is $\partial-$orthogonal to $H_0^1(\Omega)$. This may be expressed as

$$H^1(\Omega) \ = \ H_0^1(\Omega) \oplus_\partial \mathcal{H}(\Omega), \tag{3.2}$$

where $\oplus_\partial$ indicates that this is a $\partial-$orthogonal decomposition. This result is also discussed in section 22.4 of Zeidler [21].

A non-zero function $s \in H^1(\Omega)$ is said to be a *harmonic Steklov eigenfunction* on Ω corresponding to the Steklov eigenvalue δ provided s satisfies

$$\int_\Omega \nabla s \cdot \nabla v\ dx \ = \ \delta \int_{\partial\Omega} s\, v\ \ d\tilde{\sigma} \qquad \text{for all} \quad v \in H^1(\Omega). \tag{3.3}$$

This is the weak form of the boundary value problem

$$\Delta\, s \ = \ 0 \qquad \text{on } \Omega \text{ with} \quad D_\nu\, s \ = \ \delta\ |\partial\Omega|^{-1}\ s \quad \text{on } \partial\Omega. \tag{3.4}$$

Here Δ is the Laplacian and $D_\nu\, s(x) := \nabla s(x) \cdot \nu(x)$ is the unit outward normal derivative of s at a point on the boundary.

$\delta_0 = 0$ is the least eigenvalue of this problem corresponding to the eigenfunction $s_0(x) \equiv 1$ on Ω. This eigenvalue is simple as Ω is connected. All other eigenvalues of (3.3) are strictly positive.

These eigenvalues and a corresponding family of $\partial-$orthonormal eigenfunctions may be found using variational principles as described in sections 6 and 7 of Auchmuty [3]. Let the first k Steklov eigenvalues be $0 = \delta_0 < \delta_1 \leq \delta_2 \leq \ldots \leq \delta_{k-1}$ and $s_0, s_1, \ldots, s_{k-1}$ be a corresponding set of $\partial-$orthonormal eigenfunctions. The k-th eigenfunction s_k will be a maximizer of the functional

$$\mathcal{B}(u) \; := \; \int_{\partial\Omega} |\Gamma u|^2 \; d\tilde{\sigma} \tag{3.5}$$

over the subset B_k of functions in $H^1(\Omega)$ which satisfy

$$\|u\|_{\partial} \; \leq \; 1 \qquad \text{and} \qquad \langle \Gamma u, \Gamma s_l \rangle_{\partial\Omega} \; = \; 0 \quad \text{for} \quad 0 \leq l \leq k-1. \tag{3.6}$$

The existence and some properties of such eigenfunctions are described in sections 6 and 7 of [3] for a more general system. In particular, that analysis shows that each δ_j is of finite multiplicity and $\delta_j \to \infty$ as $j \to \infty$; see Theorem 7.2 of [3]. The maximizers not only are $\partial-$orthonormal but they also satisfy

$$\int_{\Omega} \nabla s_k \cdot \nabla s_l \; dx \; = \; \int_{\partial\Omega} s_k \, s_l \; d\tilde{\sigma} \; = \; 0 \qquad \text{for} \qquad k \neq l. \tag{3.7}$$

$$\int_{\Omega} |\nabla s_k|^2 \; dx \; = \; \frac{\delta_k}{1+\delta_k} \qquad \text{and} \qquad \int_{\partial\Omega} |\Gamma s_k|^2 \; d\tilde{\sigma} \; = \; \frac{1}{1+\delta_k} \quad \text{for} \quad k \geq 0. \tag{3.8}$$

Recently Daners [8] corollary 4.3 has shown that, when Ω is a Lipschitz domain, then the Steklov eigenfunctions are continuous on $\overline{\Omega}$.

Let $\mathcal{S} := \{s_j \, : \, j \geq 0\}$ be the maximal family of $\partial-$orthonormal eigenfunctions constructed inductively as above. Given $u \in H^1(\Omega)$, consider the series

$$P_H \, u(x) \; := \; \sum_{j=0}^{\infty} \; [u, s_j]_{\partial} \; s_j(x). \tag{3.9}$$

Theorem 3.1. *Assume $\Omega, \partial\Omega$ satisfy (B), then $\mathcal{S}$ is an orthonormal basis of $\mathcal{H}(\Omega)$. P_H defined by (3.9), is the $\partial-$orthogonal projection of $H^1(\Omega)$ onto $\mathcal{H}(\Omega)$.*

Proof. This follows from standard results about orthogonal expansions and theorem 7.3 of [3] which says that $\mathcal{S}$ is a maximal orthonormal subset of $\mathcal{H}(\Omega)$. □

Since $\mathcal{S}$ is a basis of $\mathcal{H}(\Omega)$, the Riesz-Fischer theorem implies that a series

$$u(x) \; := \; \sum_{j=0}^{\infty} \; c_j \, s_j(x) \tag{3.10}$$

represents a finite energy harmonic function on Ω if and only if

$$\sum_{j=0}^{\infty} |c_j|^2 \; < \; \infty. \tag{3.11}$$

Such series will be called a *harmonic Steklov expansion* and c_j is the *j-th Steklov coefficient.*

4. Hilbert spaces $\mathcal{H}^s(\Omega)$ of Real harmonic Functions on Ω.

The space $\mathcal{H}(\Omega)$ defined in the preceding section is the class of all H^1- harmonic functions on Ω. The standard convention has been to characterize the boundary values of such functions on smooth regions as being $H^{1/2}(\partial\Omega)$. Recently Auchmuty [4] described an intrinsic theory of trace spaces $H^s(\partial\Omega)$ for arbitrary real s using Steklov expansions. This can be done on more general regions than the analysis, using Fourier transforms and mappings from a half-space, of the trace spaces described by Lions and Magenes [19], McLean [20] or other recent texts.

Here Steklov series expansions and these intrinsic trace spaces will be used to define a family $\mathcal{H}^s(\Omega)$ of Hilbert spaces of harmonic functions on Ω. The classes are characterized by the regularity of their boundary traces. While the notation, and the conceptual definition, is similar to that of J-L Lions' in [18], the spaces are defined differently and may be somewhat different. The definition used here was previously described in Auchmuty and Kloucek [5] where these spaces were used to describe the solutions of some boundary value problems with discontinuous Dirichlet data that arise in the theory of thin films.

Let $\mathcal{H}_F(\Omega)$ be the class of all finite linear combinations of functions in $\mathcal{S}$, then

$$\mathcal{H}_F(\Omega) \;\subseteq\; \quad C^\infty(\Omega) \cap H^1(\Omega).$$

From theorem 3.1, $\mathcal{H}(\Omega)$ is the completion of $\mathcal{H}_F(\Omega)$ with respect to the topology induced by the $\partial-$norm.

Given $s \in \mathbb{R}$ and $u \;=\; \sum_{j=0}^{\infty} c_j s_j(x)$ in $\mathcal{H}_F(\Omega)$ define the *s-norm* of u by

$$\|u\|_s^2 \;:=\; \sum_{j=0}^{\infty} (1+\delta_j)^{2(s-1)} \; c_j^2. \tag{4.1}$$

Let $\mathcal{H}^s(\Omega)$ be the completion of the space $\mathcal{H}_F(\Omega)$ with respect to this s-norm

When u, v are Steklov expansions in $\mathcal{H}^s(\Omega)$ with coefficients $\{c_j, \; d_j\}$ respectively, consider the *s-inner product* defined by

$$\langle u, v\rangle_s \quad := \quad \sum_{j=0}^{\infty} (1+\delta_j)^{2(s-1)} \; c_j \; d_j. \tag{4.2}$$

With this inner product, $\mathcal{H}^s(\Omega)$ is a real Hilbert space. $\mathcal{S}$ will be an orthogonal set in $\mathcal{H}^s(\Omega)$ and

$$\mathcal{S}_s \;:=\; \{k_j \, s_j \;\; : j \geq 0\} \qquad \text{with} \quad k_j \;:=\; (1\,+\,\delta_j)^{1-s}$$

is an orthonormal basis of $\mathcal{H}^s(\Omega)$.

Observe that $\mathcal{H}^1(\Omega) = \mathcal{H}(\Omega)$ and $\mathcal{H}^s(\Omega)$ is a dense subspace of the space $\mathcal{H}(\Omega)$ for $s > 1$. More generally the following holds.

Theorem 4.1. *Assume that $\Omega, \partial\Omega$ satisfy (B) and $\mathcal{H}^s(\Omega)$ is defined as above. If $1 \le s_1 < s_2$, then $\mathcal{H}^{s_2}(\Omega)$ is a dense subspace of $\mathcal{H}^{s_1}(\Omega)$ and the imbedding of $\mathcal{H}^{s_2}(\Omega)$ into $\mathcal{H}^{s_1}(\Omega)$ is compact.*

Proof. For $M \ge 1$, let $P_M : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ be the finite rank operator corresponding to the M-th partial sum of the Steklov expansion (3.9). That is

$$P_M\, u(x) \;:=\; \sum_{j=0}^{M} \,[u, s_j]_\partial \; s_j(x) \qquad \text{for } u \in \mathcal{H}(\Omega). \tag{4.3}$$

Obviously $P_M\, u \in \mathcal{H}^s(\Omega)$ for all $s \ge 1$ and the definition (4.1) yields that

$$\|u\|_{s_1} \;\le\; \|u\|_{s_2} \qquad \text{whenever} \quad 1 \le s_1 < s_2. \tag{4.4}$$

Given $u \in \mathcal{H}^{s_1}(\Omega)$, the sequence $\{P_M\, u : M \ge 1\}$ is a subset of $\mathcal{H}^{s_2}(\Omega)$ which converges to u in $\mathcal{H}^{s_1}(\Omega)$, so $\mathcal{H}^{s_2}(\Omega)$ is a dense subspace of $\mathcal{H}^{s_1}(\Omega)$.

Consider the linear map $L_\theta : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ defined by

$$L_\theta\, u(x) \;:=\; \sum_{j=0}^{\infty} \,(1+\delta_j)^{-\theta}\; [u, s_j]_\partial \; s_j(x). \tag{4.5}$$

For $\theta > 0,\ L_\theta$ is a compact linear operator as it may be uniformly approximated by a finite rank operator, since $\delta_j \to \infty$, . Moreover

$$\|L_\theta\, u\|_s^2 \;=\; \sum_{j=0}^{\infty} \,(1+\delta_j)^{2(s-1-\theta)}\; [u, s_j]_\partial^2. \tag{4.6}$$

Thus L_θ is a linear isometry of $\mathcal{H}(\Omega)$ onto $\mathcal{H}^{1+\theta}(\Omega)$ so the imbedding of $\mathcal{H}^s(\Omega)$ into $\mathcal{H}(\Omega)$ is compact whenever $s > 1$. A translation in s, then yields that the imbedding of $\mathcal{H}^{s_2}(\Omega)$ into $\mathcal{H}^{s_1}(\Omega)$ is compact whenever $1 \le s_1 < s_2$. $\square$

The family of spaces $\mathcal{H}^s(\Omega)$ with $s \ge 1$ form an interpolatory family (or scale) of real Hilbert spaces as these s-norms satisfy the following log-convexity inequality.

Theorem 4.2. *Assume that $\Omega, \partial\Omega$ satisfy (B) and $\mathcal{H}^s(\Omega)$ is defined as above. If $1 \le s_1 < s_2$ and $s = (1-\theta)s_1 + \theta s_2$ with $0 \le \theta \le 1$, then*

$$\|u\|_s \;\le\; \|u\|_{s_1}^{1-\theta}\; \|u\|_{s_2}^{\theta} \qquad \textit{for all} \quad u \in \mathcal{H}^{s_2}(\Omega). \tag{4.7}$$

Proof. This is obviously true when $\theta = 0$ or 1. Assume $0 < \theta < 1$, then from (4.1),

$$\|u\|_s^2 \;:=\; \sum_{j=0}^{\infty} \,(1+\delta_j)^{2s}\, {c_j}^2. \tag{4.8}$$

Factor each term in the sum, so that $(1+\delta_j)^{2s}\, {c_j}^2 \;=\; a_j\, b_j$ with

$$a_j := (1+\delta_j)^{s_1(1-\theta)}\, {c_j}^{2(1-\theta)}, \qquad b_j := (1+\delta_j)^{s_2\theta}\, {c_j}^{2\theta}.$$

Apply Holder's inequality to (4.8) with $p := 1/(1-\theta), p^* := \theta^{-1}$, then (4.7) follows. $\square$

5. Duality for Spaces of Harmonic Functions.

For $s < 1$, $\mathcal{H}^s(\Omega)$ will include some *generalized harmonic functions* that are not H^1- (or weak) solutions of Laplace's equation. These "harmonic functions" may be interpreted in some different ways. One is in terms of the dual space of $\mathcal{H}^s(\Omega)$ with $s \geq 1$.

Given $f \in \mathcal{H}(\Omega)$ the linear functional F on $\mathcal{H}^s(\Omega)$ defined by

$$F(u) \;=\; [u,f]_\partial \;=\; \int_\Omega \nabla u \cdot \nabla f \; dx \;+\; \int_{\partial\Omega} u\; f \;\; d\tilde{\sigma} \tag{5.1}$$

is continuous for $s \geq 1$ from Schwarz' inequality and theorem 4.1. When f, u have Steklov expansions with Steklov coefficients f_j, c_j, this becomes

$$F(u) \;=\; \sum_{j=0}^{\infty} f_j\, c_j \tag{5.2}$$

This leads to the following description of the dual space of $\mathcal{H}^s(\Omega)$ with respect to the pairing from (5.2).

Theorem 5.1. *Assume that $\Omega, \partial\Omega$ satisfy (B), and F is a continuous linear functional on $\mathcal{H}^{1+\theta}(\Omega)$ with $\theta > 0$. Then there is a unique generalized harmonic function $f \in \mathcal{H}^{1-\theta}(\Omega)$ with Steklov coefficients f_j such that (5.2) holds. Moreover the dual norm of F is $\|f\|_{1-\theta}$.*

Proof. Let $s = 1+\theta$ and rewrite each term in the sum (5.2) as the product of

$$a_j := \mu_j^{-1}\, f_j \quad \text{and} \quad b_j := \mu_j\, c_j \qquad \text{with} \quad \mu_j := (1+\delta_j)^{\theta}.$$

Apply Schwarz' inequality to (5.2), then the definitions of the norms yield

$$|F(u)| \;\leq\; \|f\|_{1-\theta}\; \|u\|_s \tag{5.3}$$

Moreover equality holds here whenever $f_j = (1+\delta_j)^2\, g_j$ for all $j \geq 0$. Since F is continuous if and only if it is bounded, we see that each continuous linear functional on $\mathcal{H}^s(\Omega)$ will be represented by a generalized function in $\mathcal{H}^{1-\theta}(\Omega)$. The dual norm is defined by

$$\|F\|_{*s} \;:=\; \sup_{\|u\|_s \;\leq\; 1} \quad |F(u)|$$

which equals the norm on $\mathcal{H}^{1-\theta}(\Omega)$ from the fact that equality is attained in (5.3). $\square$

This shows that the generalized harmonic functions in $\mathcal{H}^s(\Omega)$ with $s < 1$ may be regarded as linear functionals on subspaces of the space $\mathcal{H}(\Omega)$ with respect to the pairing induced by the $\partial-$ inner product on $\mathcal{H}(\Omega)$.

6. Dirichlet Problems for Harmonic Functions.

Assume Ω is a region in $\mathbb{R}^n$ which satisfies (B). There are a variety of different theories for the solvability of Dirichlet problems for Laplace's equation on Ω. See the comprehensive survey by Benilan in chapter 2 of [9] and also the lectures of Kenig [16]. Here a Hilbert space theory that uses Steklov expansions will be described and some regularity results obtained.

Let Γ be the trace mapping as in section 2 and define

$$\hat{s_j}(y) \ := \ \sqrt{1+\delta_j}\ \Gamma\, s_j(y) \qquad \text{for } y \in \partial\Omega, \ \text{ and } \ j \geq 0. \tag{6.1}$$

Then (3.7) - (3.8) and theorem 3.1 imply that the set $\hat{\mathcal{S}} := \{\hat{s_j} : j \geq 0\}$ will be an orthonormal set in $L^2(\partial\Omega,\, d\sigma)$ with respect to the inner product defined in section 2.

The following result, which is theorem 4.1 in [4] provides an explicit expression for the trace operator in terms of the harmonic Steklov expansion of a function $u \in H^1(\Omega)$.

Theorem 6.1. *Assume $\Omega, \partial\Omega$ satisfy (B), with $\Gamma, \hat{\mathcal{S}}$ as above. Then $\hat{\mathcal{S}}$ is a maximal orthonormal set in $L^2(\partial\Omega,\, d\sigma)$ and for each $u \in H^1(\Omega)$,*

$$\Gamma u(y) \ = \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{-1/2}\ [u, s_j]_{\partial}\ \hat{s_j}(y) \quad \textit{for } \sigma \textit{ a.e. } y \in \partial\Omega. \tag{6.2}$$

When $g \in L^2(\partial\Omega,\, d\sigma)$, we shall call the usual expansion of g with respect to the orthonormal basis $\hat{\mathcal{S}}$, a *boundary Steklov expansion* of the boundary function. Then

$$g(y) \ = \ \sum_{j=0}^{\infty} \ g_j\ \hat{s_j}(y) \qquad \text{for } \ y \in \partial\Omega \quad \text{with } g_j \ = \ \langle g, \hat{s_j}\rangle_{\partial\Omega}. \tag{6.3}$$

The space $H^s(\partial\Omega)$ for $s \geq 0$ is defined to be the subspace of functions $g \in L^2(\partial\Omega,\, d\sigma)$ with boundary Steklov expansion (6.3), satisfying

$$\sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, {g_j}^2 \ < \ \infty. \tag{6.4}$$

The *s-inner product* and *s-norm* on $H^s(\partial\Omega)$ are defined by

$$[g,h]_{s,\partial\Omega} \ := \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, g_j\, h_j \qquad \text{and} \qquad \|g\|^2_{s,\partial\Omega} \ := \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{2s}\, {g_j}^2. \tag{6.5}$$

See [4] for more details about these spaces and their properties.

When $s = 0$, then $H^0(\partial\Omega) = L^2(\partial\Omega,\, d\sigma)$. When $s = 1/2$, define the map $E : H^{1/2}(\partial\Omega) \to \mathcal{H}(\Omega)$ by

$$E\, g(x) \ := \ \sum_{j=0}^{\infty} \ (1+\delta_j)^{1/2} \ \ g_j\, s_j(x). \tag{6.6}$$

E may be regarded as the *harmonic extension* of the function g defined on $\partial\Omega$ to the region Ω since $E\,g$ is a harmonic function on Ω with $\Gamma E\,g = g$ from (6.1). That is, the operator E is the Steklov representation of the usual Poisson operator associated with the Dirichlet problem for Laplace's equation. Classically, the harmonic function on Ω associated with boundary trace g is written as

$$u(x) \;:=\; \int_{\partial\Omega} P(x,y)\,g(y)\,d\tilde{\sigma}. \tag{6.7}$$

Comparing (6.7) with (6.6) and (6.1) leads to the formula

$$P(x,y) \;=\; \sum_{j=0}^{\infty} (1+\delta_j)\,s_j(x)\,\Gamma s_j(y) \qquad \text{for } (x,y) \in \Omega\times\partial\Omega. \tag{6.8}$$

This is the Steklov expansion of the Poisson kernel on a general domain and generalizes the well-known formulae for balls in $\mathbb{R}^n$ in terms of spherical harmonics.

Theorem 6.2. *Assume $\Omega, \partial\Omega$ satisfy (B), with $\Gamma, \hat{\mathcal{S}}$ as above. Then E is an isometric isomorphism of $H^{1/2}(\partial\Omega)$ and $\mathcal{H}(\Omega)$*

Proof. When $g \in H^{1/2}(\partial\Omega)$ has the bundary Steklov expansion (6.3) then, from (6.5),

$$\|g\|^2_{1/2,\partial\Omega} \;=\; \sum_{j=0}^{\infty}\,(1+\delta_j)\,{g_j}^2$$

Substitute (6.6) in the formula (4.1) for the $1-$ norm on $\mathcal{H}(\Omega)$. Since $c_j = \sqrt{(1+\delta_j)}\,g_j$ we see that $\|E\,g\|_1 \;=\; \|g\|_{1/2,\partial\Omega}$ for each g so E is an isometry. Since $\mathcal{S}, \hat{\mathcal{S}}$ are bases of the respective spaces, E is a linear isomorphism. □

This result agrees formally with the classical results for the finite energy Dirichlet problem when $\partial\Omega$ is a smooth manifold. This result extends to general $s \in \mathbb{R}$. Consider the map $E_s : H^s(\partial\Omega) \to \mathcal{H}^s(\Omega)$ defined by

$$E_s\,g(x) \;:=\; \sum_{j=0}^{\infty}\,(1+\delta_j)^{1/2}\;\,g_j\,s_j(x). \tag{6.9}$$

The following general result holds.

Theorem 6.3. *Assume $\Omega, \partial\Omega$ satisfy (B), with $\Gamma, \hat{\mathcal{S}}$ as above. Given $s \in \mathbb{R}$, E_s is an isometric isomorphism of $H^s(\partial\Omega)$ and $\mathcal{H}^{s+1/2}(\Omega)$.*

Proof. Just as in the previous theorem, $g \in H^s(\partial\Omega)$ has

$$\|g\|^2_{s,\partial\Omega} \;=\; \sum_{j=0}^{\infty}\,(1+\delta_j)^{2s}\,{g_j}^2 \;=\; \|E_s\,g\|^2_{s+1/2}$$

so the result follows. □

This result shows that the Hilbert spaces $\mathcal{H}^{s+1/2}(\Omega)$ of real harmonic functions on Ω are determined precisely by the corresponding boundary trace spaces $H^s(\partial\Omega)$. Essentially $\Gamma\, E_s$ is the identity map on $H^s(\partial\Omega)$, and $E_s\,\Gamma$ is the identity map on $\mathcal{H}^{s+1/2}(\Omega)$. In the following the subscript s may be omitted when no confusion is likely and E_s will be called a *harmonic extension operator*.

While the notation might suggest that $\mathcal{H}^s(\Omega)$ is the subspace of harmonic functions in $H^s(\Omega)$, the author does not know whether this holds for $s \neq 1$, or under what conditions on $\partial\Omega$ this might be true.

There are weaker criteria for solutions of Laplace's equation. A function $u \in L^2(\Omega)$ is an *ultra-weak solution* of Laplace's equation on Ω provided it satisfies

$$\int_\Omega u\ \Delta\varphi\ dx \;=\; 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{6.10}$$

Let $\mathcal{H}L^2(\Omega)$ be the subspace of $L^2(\Omega)$ of all ultra-weak solutions of Laplace's equation. It is a closed subspace of the real Hilbert space $L^2(\Omega)$.

It would also be of considerable interest to know the relationship of $\mathcal{H}L^2(\Omega)$ and $\mathcal{H}^s(\Omega)$ under various regularity conditions on $\partial\Omega$. The analysis in [16] chapter 1, indicates that if Ω is a Lipschitz region and u is in $\mathcal{H}^s(\Omega)$ for some $s \geq 1/2$, then $u \in \mathcal{H}L^2(\Omega)$. Essentially $u \in \mathcal{H}^s(\Omega)$ with $s = 1/2$ if and only if $\Gamma u \in L^2(\partial\Omega,\, d\sigma)$. When Ω is a Lipschitz domain then $g \in L^2(\partial\Omega,\, d\sigma)$ implies $Eg \in L^2(\Omega)$ so $u := Eg$ is in $\mathcal{H}L^2(\Omega)$. The natural conjecture is that for nice enough regions Ω, $\mathcal{H}L^2(\Omega) = \mathcal{H}^0(\Omega)$ but the author does not know of a proof of this

7. Pointwise Bounds for Harmonic Functions.

A real Hilbert space H of functions on a region Ω is said to be a *reproducing kernel Hilbert space* (RKHS) provided the linear functional $\delta_x : H \to \mathbb{R}$ defined by $\delta_x(u) \;:=\; u(x)$ is continuous for each $x \in \Omega$. Even when Ω is a planar disc, there are elementary examples of functions in $\mathcal{H}(\Omega)$ which are unbounded on $\partial\Omega$. Here we shall show that for $s \geq 1/2$, $\mathcal{H}^s(\Omega)$ is an RKHS with respect to Ω.

To do this, variational methods will be used to obtain some optimal inequalities for the pointwise behavior of harmonic functions. The results will first be obtained for $\mathcal{H}(\Omega)$ and then the same issues will be studied for $\mathcal{H}L^2(\Omega)$. Note that we only seek continuity on Ω - not on $\overline{\Omega}$.

Suppose $x^{(0)} \in \Omega$ and $d_0 := d(x^{(0)}, \partial\Omega) > 0$. Given $0 < r < d_0$, define the linear functional $\chi_r : \mathcal{H}(\Omega) \to \mathbb{R}$ and the subset B_1 by

$$\chi_r(u) \;:=\; \int_{|x-x^{(0)}|\leq r} a_r\ u\ dx \qquad \text{and} \quad B_1 \;:=\; \{u \in \mathcal{H}(\Omega) : \|u\|_\partial \leq 1\}. \tag{7.1}$$

Here $a_r = nr^n/\omega_n$ is a constant chosen so that $\chi_r(1) = 1$; ω_n is the surface area of the unit ball in $\mathbb{R}^n$. Consider the variational problem of maximizing χ_r on the set B_1 and define

$$c_1(r) := \sup_{u \in B_1} \chi_r(u). \tag{7.2}$$

The following elementary result enables the proof that $\mathcal{H}(\Omega)$ is a RKHS.

Lemma 7.1. *Assume $\Omega, x^{(0)}, d_0, r$ as above and (B) holds, then c_1 is finite, there is a maximizer of χ_r on B_1 and*

$$|u(x^{(0)})| \leq c_1 \|u\|_\partial \qquad \text{for all } u \in \mathcal{H}(\Omega). \tag{7.3}$$

Proof. χ_r is a weakly continuous functional on $H^1(\Omega)$ and B_1 is bounded, convex and weakly closed. Thus it is weakly compact so c_1 is finite and the supremum is attained. When $u \in \mathcal{H}(\Omega), x^{(0)} \in \Omega$, then u is C^∞ on the r-ball centered at $x^{(0)}$ for r as above so the mean value theorem for harmonic functions implies that $u(x^{(0)}) = \chi_r(u)$ and (7.3) holds. □

Corollary 7.2. *Assume (B) holds and $s \geq 1$, then $\mathcal{H}^s(\Omega)$ is a RKHS.*

Proof. If $\delta_{x^{(0)}}$ is in the dual space of $\mathcal{H}(\Omega)$, then it is in the dual space of $\mathcal{H}^s(\Omega)$ for all $s \geq 1$. The preceding result shows that $\delta_{x^{(0)}}$ is a continuous linear functional on $\mathcal{H}(\Omega)$. □

It is worth noting that $c_1(r)$ is dependent implicitly on the region Ω via the definition of the $\partial-$inner product. To treat the case of $\mathcal{H}L^2(\Omega)$, consider the problem of maximizing $\chi_r : L^2(\Omega) \to \mathbb{R}$ defined as in (7.1) on the set

$$B_2 := \{u \in \mathcal{H}L^2(\Omega) : \int_\Omega u^2 \, dx \leq 1\}. \qquad \text{Let} \quad c_2(r) := \sup_{u \in B_2} \chi_r(u). \tag{7.4}$$

Lemma 7.3. *Assume $\Omega, x^{(0)}, d_0, r$ as above and (B) holds, then c_2 is finite, there is a maximizer of χ_r on B_2 and*

$$|u(x^{(0)})| \leq c_2 \|u\|_2 \qquad \text{for all } u \in \mathcal{H}L^2(\Omega). \tag{7.5}$$

Proof. χ_r is a weakly continuous functional on $L^2(\Omega)$ and B_2 is bounded, convex and weakly closed. Thus it is weakly compact so c_2 is finite and the supremum is attained. When $u \in \mathcal{H}L^2(\Omega),\ x^{(0)} \in \Omega$, then u is C^∞ on the r-ball centered at $x^{(0)}$ for r as above so the mean value theorem for harmonic functions implies that $u(x^{(0)}) = \chi_r(u)$ and (7.4) holds. □

The following result is an immediate consequence of this lemma and the comments at the end of the previous section.

Corollary 7.4. *Assume (B) holds then $\mathcal{H}L^2(\Omega)$ is a RKHS and $\mathcal{H}^s(\Omega)$ is a RKHS for $s \geq 1/2$.*

As commented in the introduction, it would be interesting to find the least value of s for which $\mathcal{H}^s(\Omega)$ is a RKHS and whether there are generalizations of the inequality (7.3) valid for general s, when the boundary $\partial\Omega$ is smooth enough.

8. Reproducing Kernels for $\mathcal{H}^s(\Omega)$.

Suppose that a real Hilbert space H is a RKHS, then the Riesz-Fr/'echet characterization of continuous linear functionals says that there is a map $R : \Omega \to H$ with

$$\delta_x(u) \;=\; u(x) \;=\; \langle R(x), u\rangle_H \qquad \text{for each } x \in \Omega \text{ and } \; u \in H. \tag{8.1}$$

This map R is called the reproducing kernel of H with respect to this inner product - and it depends on the chosen inner product.

Some results on reproducing formulae for solutions of second-order elliptic boundary value problems were already described in part B of Bergman and Schiffer [6] and provided considerable motivation for the original development of the theory of RKH spaces at that time. While the theory was successfully applied to important classes of complex Hilbert spaces, there appears to be only a small literature on reproducing kernels for real Hilbert spaces of solutions of partial differential equations.

In particular J-L. Lions in [17], [18] and Englis, Lukhausen, Peetre and Persson in [12] have described characterizations of the reproducing kernel on various different spaces of real harmonic functions. Their characterizations required considerable smoothness of the boundary and involve the use of classical Green's functions, or eigenfunctions of the Laplace-Beltrami operator on the boundary of the region. They also use Laplace's equation as a prototype and indicate that similar analyses could be applied to large classes of elliptic equations.

Here we shall use simple, direct methods to obtain formulae for the reproducing kernels for the spaces $\mathcal{H}^s(\Omega)$. These formulae will be explicit expansions in terms of the Steklov bases of the spaces. Since the reproducing kernel depends on the inner product for H, the formulae obtained here are quite different to the expressions obtained in [18] and [12]. Essentially the use of the $\partial-$ inner product on $\mathcal{H}(\Omega)$ and the s-inner product on $\mathcal{H}^s(\Omega)$ enables particularly simple representations of the reproducing kernel.

(a) The reproducing kernel for $\mathcal{H}(\Omega)$.

The reproducing kernel for $\mathcal{H}(\Omega)$ with respect to the inner product defined by (4.2) with $s = 1$ is the mapping $R_1 : \Omega \to \mathcal{H}(\Omega)$ that satisfies

$$u(x) \;=\; \langle R_1(x), u\rangle_1 \qquad \text{for each } x \in \Omega \text{ and } u \in \mathcal{H}(\Omega). \tag{8.2}$$

Theorem 8.1. *Assume (B) holds and $x \in \Omega$, then the reproducing kernel for $\mathcal{H}(\Omega)$ is $R_1(x) \in \mathcal{H}(\Omega)$ where*

$$R_1(x) \;:=\; \sum_{j=0}^{\infty} \; s_j(x) \; s_j \qquad \textit{for} \;\; x \in \Omega. \tag{8.3}$$

Proof. Corollary (7.2) proves that $\mathcal{H}(\Omega)$ is a RKHS, so we only need to evaluate the kernel. From theorem 3.1, $\mathcal{S}$ is a basis of $\mathcal{H}(\Omega)$ so it suffices to find the Steklov expansion of R_1. Suppose u has the expansion (3.10) and substitute (8.3) in (4.2), then

$$\langle R_1(x), u\rangle_1 \ = \ \sum_{j=0}^{\infty} c_j \ s_j(x)$$

as the s_j are orthonormal in $\mathcal{H}(\Omega)$. Thus (8.2) holds and (8.3) follows by uniqueness. $\square$

Since $R_1(x)$ is in $\mathcal{H}(\Omega)$ for each x, it may be regarded as a C^∞ function on Ω with the Steklov representation

$$R_1(x)(z) \ := \ \sum_{j=0}^{\infty} s_j(x) \ s_j(z) \qquad \text{for } (x,z) \in \Omega \times \Omega.$$

(b). The reproducing kernel for $\mathcal{H}^s(\Omega)$.

For $s \geq 1$, corollary (7.2) implies that $\mathcal{H}^s(\Omega)$ is a RKHS. The comments at the end of section 6 and corollary 7.4 imply that this also holds when $s \geq 1/2$. Whenever it is a RKHS then the reproducing kernel $R_s : \Omega \to \mathcal{H}^s(\Omega)$ will be the mapping that satisfies

$$u(x) \ = \ \langle R_s(x), u\rangle_s \qquad \text{for each } x \in \Omega \text{ and } u \in \mathcal{H}^s(\Omega). \tag{8.4}$$

The general formula for the reproducing kernel is the following..

Theorem 8.2. *Assume $\mathcal{H}^s(\Omega)$ is a RKHS and $x \in \Omega$, then the reproducing kernel for $\mathcal{H}^s(\Omega)$ is given by*

$$R_s(x) \ := \ \sum_{j=0}^{\infty} (1+\delta_j)^{2(1-s)} \ s_j(x) \ s_j \qquad \text{for each } x \in \Omega \tag{8.5}$$

Proof. Suppose u has the expansion (3.10) and substitute (8.5) in (4.2), then

$$\langle R_s(x), u\rangle_s \ = \ \sum_{j=0}^{\infty} c_j \ s_j(x)$$

which is (8.4) as required. $\square$

Comparison of this expression with (6.7) - (6.8) shows that the Poisson kernel $P(x,.)$ can be regarded as the boundary trace of this reproducing kernel for $\mathcal{H}^{1/2}(\Omega)$. Alternatively the reproducing kernel $R_{1/2}(x)$ is the harmonic extension of the Poisson kernel $P(x,.)$ regarded as a function on $\partial\Omega$.

In the next sections for special values of s, these reproducing kernels, and related operators will be related to solution operators of some boundary value problems for the Laplacian in a number of ways.

9. Green's functions and the Kernel function

In the preceding works on reproducing kernels, most of the formulae have involved the solution operators for the Laplacian subject to various boundary conditions. In particular Bergman and Schiffer [6], Part B, chapter 2, focussed attention on the *kernel function* which they defined to be the difference between the Dirichlet Green's function and the Neumann Green's function for an elliptic operator.

Here we shall show that the linear integral operator associated with their kernel function can be expressed in terms of the harmonic projection operator P_H defined in section 3 and the Neumann Green's function. In particular it may be represented using a Steklov series expansion.

Denote by $L^2_m(\Omega), H^1_m(\Omega)$ the closed subspaces of $L^2(\Omega), H^1(\Omega)$ respectively of functions with integral 0 on Ω. By elementary variational arguments, given $\rho \in L^2_m(\Omega)$, there is a unique solution $\hat{u}$ in $H^1_m(\Omega)$ of the equation

$$\int_\Omega \nabla u \cdot \nabla v \; dx \;=\; \int_\Omega \rho \, v \; dx \qquad \text{for all } v \in H^1(\Omega). \tag{9.1}$$

Denote this solution as $\hat{u}(x) \;:=\; G_N \, \rho(x)$ where $G_N : L^2_m(\Omega) \to H^1_m(\Omega)$ is the Neumann Green's operator. It is a continuous linear transformation.

Similarly let $G_D : L^2_m(\Omega) \to H^1_0(\Omega)$ be the operator for which $\tilde{u} := G_D \rho$ is the solution in $H^1_0(\Omega)$ of the system

$$\int_\Omega \nabla u \cdot \nabla v \; dx \;=\; \int_\Omega \rho \, v \; dx \qquad \text{for all } v \in H^1_0(\Omega) \tag{9.2}$$

Let $w := \hat{u} - \tilde{u}$, then $w \in H^1(\Omega)$ satisfies (3.1) so it is weakly harmonic on Ω. Define the *kernel operator* $K : L^2_m(\Omega) \to H^1(\Omega)$ by

$$K\rho(x) \;:=\; [G_N \,-\, G_D]\, \rho(x) \tag{9.3}$$

as in [6], Part B, chapter 2, then the above analysis and (3.2) proves the following result.

Theorem 9.1. *Assume that (B) holds and K, G_D, G_N and P_H are defined as above then*

$$K\rho(x) \;=\; P_H \, G_N \, \rho(x), \qquad \textit{for all } \rho \in L^2_m(\Omega).$$

In particular this theorem shows that the range of the kernel operator is a subspace of $\mathcal{H}(\Omega)$. Moreover $K\rho$ has a Steklov expansion of the form (3.9) for $\rho \in L^2_m(\Omega)$. A (60 page) chapter of [6] is devoted to this operator and its properties while the operator and some of its properties are also described in Garabedian's well-known text [14]

10. Integral kernels for Robin and Neumann problems

Here we will describe Steklov expansions for weak solutions of Robin and Neumann boundary value problems for Laplace's equation. These formulae may be used to describe the analogues of the Poisson kernel for these solutions.

When $g \in L^2(\partial\Omega,\, d\sigma)$, we say that a function $\tilde{u} \in \mathcal{H}(\Omega)$ is a weak solution of Laplace's equation subject to the Robin boundary condition

$$D_\nu\, u(y) + b\, u(y) \;=\; g(y) \qquad \text{for } y \in \partial\Omega \tag{10.1}$$

provided $\tilde{u}$ satisfies the system

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} (b\, u - g)\, v \;\, d\sigma \;=\; 0 \qquad \text{for all } v \in H^1(\Omega). \tag{10.2}$$

Here $b \in (0, \infty)$; analogous results will hold when b is a non-zero, positive L^∞ function on $\partial\Omega$ - but then the appropriate weighted harmonic Steklov eigenfunctions associated with this b must be used as described in [3].

When $g \in L^2(\partial\Omega,\, d\sigma)$ has the Steklov representation

$$g(y) \;=\; \sum_{j=0}^{\infty} g_j \;\hat{s}_j(y) \qquad \text{for } \; y \in \partial\Omega \quad \text{with } g_j \;=\; \langle g, \hat{s}_j \rangle_{\partial\Omega} \tag{10.3}$$

then the solution of (10.2) has the Steklov expansion

$$\tilde{u}(x) \;=\; |\partial\Omega| \sum_{j=0}^{\infty} \frac{(1+\delta_j)^{1/2}}{b|\partial\Omega| + \delta_j} \; g_j \; s_j(x) \qquad \text{for } x \in \Omega. \tag{10.4}$$

This result could also be written as

$$\tilde{u}(x) \quad := \quad \int_{\partial\Omega} P_b\,(x, y)\, g(y)\, d\sigma \qquad \text{where} \tag{10.5}$$

$$P_b\,(x, y) \quad = \quad \sum_{j=0}^{\infty} \frac{(1+\delta_j)}{b|\partial\Omega| + \delta_j}\, s_j(x)\, \Gamma s_j(y) \qquad \text{for } (x, y) \in \Omega \times \partial\Omega. \tag{10.6}$$

As a consequence we have the following regularity result for the solutions of these Robin boundary value problems.

Theorem 10.1. *Assume $\Omega, \partial\Omega$ satisfy (B) and $b > 0$. If $g \in H^s(\partial\Omega)$ for some $s \geq 0$, then the unique solution $\tilde{u}$ of (10.2) is in $\mathcal{H}^{s+3/2}(\Omega)$ and has the representation (10.4) or (10.5)-(10.6).*

Proof. The representations were derived above by substituting s_j for v in (10.2). To show that $\tilde{u} \in \mathcal{H}^{s+3/2}(\Omega)$, the coefficients of g_j^2 in the expression for the norm must be bounded. This is straightforward and the result follows. $\square$

One may note that in the special case when $b \;=\; |\partial\Omega|^{-1}$, this function P_b is the boundary trace of the reproducing kernel R_1 for the Hilbert space $\mathcal{H}(\Omega)$ given by (8.5) with $s = 1$. That is the reproducing kernel $R_1(x)$ for $\mathcal{H}(\Omega)$ is the harmonic extension of this $P_b(x, .)$ to the region Ω.

When $b = 0$ in the above analysis, we have the Neumann harmonic boundary value problem on Ω. Suppose $L^2_m(\partial\Omega,\, d\sigma)$ is the space of functions in $L^2(\partial\Omega,\, d\sigma)$ with integral

0 on $\partial\Omega$. Given $g \in L^2(\partial\Omega,\, d\sigma)$, a function $u \in \mathcal{H}(\Omega)$ is said to be a weak solution of the Neumann problem for Laplace's equation on Ω provided it satisfies

$$\int_\Omega \nabla u \cdot \nabla v \; dx \;=\; \int_{\partial\Omega} g\, v \;\; d\sigma \qquad \text{for all } v \in H^1(\Omega). \tag{10.7}$$

A necessary condition for this problem to have a solution is that $g \in L^2_m(\partial\Omega,\, d\sigma)$. Then the solution is unique up to a constant. When $g \in L^2_m(\partial\Omega,\, d\sigma)$ has the Steklov representation (10.3) with $g_0 = 0$, then the general solution of (10.7) is

$$\tilde{u}(x) \;=\; k_0 \,+\, |\partial\Omega| \, \sum_{j=1}^{\infty} \, {\delta_j}^{-1} \, (1+\delta_j)^{1/2} \; g_j \;\, s_j(x) \qquad \text{for } \; x \in \Omega. \tag{10.8}$$

Here k_0 is an arbitrary constant. The corresponding integral representation is

$$\tilde{u}(x) \quad := \quad k_0 \,+ \int_{\partial\Omega} P_0\,(x,y)\, g(y)\, d\sigma \qquad \text{where} \tag{10.9}$$

$$P_0\,(x,y) \quad := \quad \sum_{j=1}^{\infty} \, {\delta_j}^{-1}(1+\delta_j)\, s_j(x)\, \Gamma s_j(y) \qquad \text{for } (x,y) \in \Omega \times \partial\Omega. \tag{10.10}$$

These yield the following regularity result.

Theorem 10.2. *Assume $\Omega, \partial\Omega$ satisfy (B), $s \geq 0$ and $g \in H^s(\partial\Omega)$ has integral zero on $\partial\Omega$. Then the solutions $\tilde{u}$ of (10.7) are in $\mathcal{H}^{s+3/2}(\Omega)$ and have the Steklov expansion (10.8) with $k_0 \in \mathbb{R}$.*

Proof. This is proved in exactly the same manner as theorem 10.1 above. ☐

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