THE STOKES BASIS FOR 3D INCOMPRESSIBLE FLOW FIELDS

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Abstract. A family of eigenfields of a specific Stokes eigenproblem is described which constitute an orthonormal basis of a Sobolev-Hilbert space of incompressible flow fields obeying no-slip boundary conditions. This basis is used to describe a representation theorem for such fields, including spectral formulae for the kinetic energy and the enstrophy. Some other properties of the vorticity and the helicity of these basis fields are derived and a simple formula for reconstructing the velocity from the vorticity is described.

1. Introduction

This paper will first develop the weak formulation of a spectral problem for the Stokes operator. A classical description of the Stokes eigenproblem is given in chapter 4 of Constantin and Foias [4]. The development there is done in the setting of a closed densely defined linear operator on the Hilbert space of $L^2$ vector fields on $\Omega$. Another version is described in section 2.6 of chapter 1 of Temam [8]. Here the Stokes eigenfields will be constructed using a direct variational characterization on a natural Sobolev-Hilbert space of vector fields.

These eigenfields will be proven to be a basis of the space of $H^1$ incompressible flow fields obeying no-slip boundary conditions. It will be called the Stokes basis. Various properties of these basis fields will be proved and spectral formulae for the energy and enstrophy will be derived. In particular the helicity of an eigenfield, and of its vorticity, is related by a simple formula and various formulae for the coefficients in spectral expansions of the field are derived and used.

2. Function Spaces and Notation

A region in $\mathbb{R}^3$ is a non-empty, connected, open subset of $\mathbb{R}^3$. Its closure is denoted by $\overline{\Omega}$ and its boundary is $\partial \Omega := \overline{\Omega} \setminus \Omega$. We generally require:

Condition B1: $\Omega$ is a bounded region in $\mathbb{R}^3$ and $\partial \Omega$ is the union of a finite number of disjoint closed $C^{1,1}$ surfaces; each surface having finite surface area.

A closed surface $\Sigma$ in space is said to be $C^{1,1}$ if it has a unique unit outward normal $\nu$ at each point and $\nu$ is a Lipschitz continuous vector field on $\Sigma$. See Girault and Raviart [7], Section 1.1. for more details on this definition. The surface area measure on $\partial \Omega$ will be denoted by $d\sigma$.

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When $u$, $v$ are vectors in $\mathbb{R}^3$, their scalar product, Euclidean norm and vector product are denoted $u \cdot v$, $|u|$, and $u \wedge v$, respectively. In the following $\langle u, v \rangle$ will denote the $L^2$–inner product and $\|u\|$ denotes the $L^2$– norm of a function or vector field on $\Omega$.

Let $H^1_0(\Omega)$ be the usual Sobolev-Hilbert space of all scalar valued functions on $\Omega$ and $H^1_0(\Omega; \mathbb{R}^3)$ be the corresponding space of vector fields. Our results are based on using a special, natural inner product on this space; namely

$$\langle u, v \rangle_{DC} := \int_\Omega [\text{curl } u \cdot \text{curl } v + \text{div } u \text{div } v] \, d^3x.$$ 

See [5] volume 3, chapter 9 for a proof that this inner product is equivalent to the usual $H^1$–inner product on $H^1_0(\Omega; \mathbb{R}^3)$.

When $v \in H^1_0(\Omega; \mathbb{R}^3)$, the divergence of $v$ is the function $\rho \in L^2(\Omega)$ which satisfies

$$\int_\Omega \rho \varphi \, d^3x = \int_\Omega u \cdot \nabla \varphi \, d^3x \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$

Here $C^\infty_c(\Omega)$ is the space of all $C^\infty$ functions on $\Omega$ with compact support. A field $v \in H^1_0(\Omega; \mathbb{R}^3)$ is said to be incompressible or divergence-free when $\rho \equiv 0$ in $L^2(\Omega)$. The subspace of all incompressible vector fields in $H^1_0(\Omega; \mathbb{R}^3)$ will be denoted $V^1_0(\Omega)$ and is a closed subspace of $H^1_0(\Omega; \mathbb{R}^3)$. It will be a real Hilbert space under the induced inner product

$$\langle v, w \rangle_C := \int_\Omega \text{curl } v \cdot \text{curl } w \, d^3x.$$ 

Fields in $V^1_0(\Omega)$ will be said to be C-orthonormal provided they are orthonormal with respect to this inner product.

When (B1) holds the Gauss-Green theorem holds on $\Omega$. Two consequences of the Gauss-Green theorem will be used repeatedly here. They say that the following hold when each of the individual integrals are finite,

$$\int_\Omega u \cdot \nabla \varphi \, d^3x = \int_{\partial \Omega} \varphi (u \cdot \nu) \, d\sigma - \int_\Omega \varphi \, \text{div } u \, d^3x,$$

$$\int_\Omega u \cdot \text{curl } v \, d^3x = \int_{\partial \Omega} v \cdot (u \wedge \nu) \, d\sigma + \int_\Omega v \cdot \text{curl } u \, d^3x.$$

We will need the following characterization of the $L^2$–orthogonal complement of $V^1_0(\Omega)$.

**Theorem 2.1.** Assume (B1) holds, $v \in H^1_0(\Omega; \mathbb{R}^3)$ and

$$\int_\Omega v \cdot w \, d^3x = 0 \quad \text{for all } w \in V^1_0(\Omega).$$

Then $v = \nabla \varphi$ where $\varphi \in H^1_0(\Omega)$, $\Delta \varphi \in L^2(\Omega)$ and $\partial \varphi / \partial \nu = 0$ on $\partial \Omega$.

**Proof.** This proof depends on the results of section 7 of Auchmuty [3] and the notation of that paper will be used. We first show that the space $C^1_0(\Omega)$ defined there is the same as $V^1_0(\Omega)$ defined here. When $v \in C^1_0(\Omega)$, then $v \in H^1_0(\Omega; \mathbb{R}^3)$ and $\text{div } v = 0$ so $v \in V^1_0(\Omega)$.
Conversely if \( v \in V_0^1(\Omega) \) then Corollary 7.4 says that the projection of \( v \) onto \( G_0^1(\Omega) \) is defined by \( P_G v = \nabla \tilde{\varphi} \) where \( \tilde{\varphi} \) minimizes

\[
\int_\Omega |\Delta \varphi|^2 \, d^3x \quad \text{over } \varphi \in X_0.
\]

The unique minimizer of this is \( \tilde{\varphi} = 0 \), so \( P_G v = 0 \) and thus \( v \in C_0^1(\Omega) \). Hence the two spaces are equal.

Now use (iii) of theorem 7.2 in [3]. It implies that for any \( v \in H_0^1(\Omega; \mathbb{R}^3) \), there is a \( \varphi \in X_0 \) such that

\[
v = \nabla \varphi + u \quad \text{with } u \in V_0^1(\Omega).
\]

Substitute this in (2.6) and take \( w = u \), then the left hand side is \( \|u\|^2 \). Hence \( u = 0 \) and then the theorem follows upon using the characterization of \( X_0 \) given there. \( \square \)

3. The Stokes Eigenproblem

The Stokes eigenproblem has been discussed in many texts including chapter 4 of [4] and chapter 1 of [8]. Here we shall describe a weak formulation in the setting of the Sobolev space \( V_0^1(\Omega) \) of incompressible vector fields.

A vector field \( v \in V_0^1(\Omega) \) is said to be an eigenfield for the Stokes eigenproblem on \( \Omega \) corresponding to an eigenvalue \( \lambda \) provided it is a non-zero solution of the equation

\[
\int_\Omega (\text{curl } v \cdot \text{curl } w - \lambda \, v \cdot w) \, d^3x = 0 \quad \text{for all } w \in V_0^1(\Omega).
\]

It is straightforward to verify that such a field is a weak version of the usual description of this eigenproblem. It may be considered as an eigenproblem for either the \( \text{curl}^2 := \text{curl}(\text{curl}) \) operator, or the vector Laplacian, on this space of incompressible fields. Putting \( w = v \) in (3.1) yields that any eigenvalue of this Stokes eigenproblem is positive.

The existence, and properties, of the eigenfields of this problem will be obtained using variational arguments. Consider the variational problem of minimizing the functional

\[
\mathcal{C}(v) := \|v\|_{C_0}^2 := \int_\Omega |\text{curl } v|^2 \, d^3x
\]

on the set

\[
B_1 := \{ \ v \in V_0^1(\Omega) : \int_\Omega |v|^2 \, d^3x = 1 \}.
\]

The set \( B_1 \) is a weakly closed set in \( V_0^1(\Omega) \) from Rellich’s theorem, so standard arguments, including a Poincare inequality, lead to the following result.

**Theorem 3.1.** Assume that (B1) holds, then there are fields \( \pm v^{(1)} \in V_0^1(\Omega) \) which minimize \( \mathcal{C} \) on \( B_1 \). They satisfy (3.1) with \( \lambda = \lambda_1 \) being the least eigenvalue of this problem and \( \lambda_1 = \mathcal{C}(v^{(1)}) > 0 \).
Knowing the least eigenvalue $\lambda_1$ and a corresponding eigenfield $v^{(1)}$, the successive eigenvalues $\{\lambda_j : j \geq 1\}$ and the corresponding eigenfields $v^{(j)}$ may be characterized inductively. For $J \geq 1$, define

$$B_{J+1} := \{ v \in V_0^1(\Omega) : \int_{\Omega} |v|^2 \, d^3x = 1, \int_\Omega v \cdot v^{(j)} \, d^3x = 0 \text{ for } 1 \leq j \leq J \}.$$  

Consider the problem of minimizing the functional $C$ defined by (3.2) on $B_{J+1}$. The following result describes the solutions of this problem.

**Theorem 3.2.** Assume that (B1) holds, then there are fields $\pm v^{(J+1)} \in V_0^1(\Omega)$ which minimize $C$ on $B_{J+1}$. They satisfy (3.1) with $\lambda = \lambda_{J+1} = C(v^{(J+1)})$. $\lambda_{J+1}$ is the least eigenvalue of this problem greater than or equal to $\lambda_j$.

The proofs of these theorems parallel the usual proofs of similar properties for the eigenvalues and eigenfunctions of the Dirichlet Laplacian on $H_0^1(\Omega)$. In fact the 2-dimensional analog of this problem is precisely such a problem.

In this construction the fields $v^{(j)}$ are $L^2-$ orthogonal from the definition of the $B_j$. Then equation (3.1) yields that

$$\int_\Omega \text{curl} \, v^{(j)} \cdot \text{curl} \, v^{(k)} \, d^3x = \lambda_j \delta_{jk} \quad \text{for } j, k \geq 1.$$  

That is the corresponding vorticities $\omega^{(j)} := \text{curl} \, v^{(j)}$ will be $L^2-$ orthogonal on $\Omega$. Write $\tilde{v}^{(j)} := \lambda_j^{-1/2} v^{(j)}$, then the set $B := \{ \tilde{v}^{(j)} : j \geq 1 \}$ will be a C-orthonormal set in $V_0^1(\Omega)$.

This leads to the following result about the sequence of eigenvalues $\sigma_S(\Omega) := \{ \lambda_j : j \geq 1 \}$ and the corresponding eigenfields of this eigenproblem.

**Theorem 3.3.** Assume that (B1) holds, and the sequences $\sigma_S(\Omega), B$ are defined as above. Then each eigenvalue $\lambda_j$ has finite multiplicity and $\lambda_j \to \infty$ as $j \to \infty$. Moreover $B$ is a maximal C-orthonormal set in $V_0^1(\Omega)$.

**Proof.** Suppose that there are infinitely many $L^2-$ orthogonal eigenfields $v^{(j)}$ of this Stokes eigenproblem corresponding to eigenvalues $\lambda_j$ with $\lambda_j < c$ for some number $c$. Then (3.1) implies that

$$\|v^{(j)}\|_{C^2}^2 = \int_\Omega |\text{curl} \, v^{(j)}|^2 \, d^3x < c \quad \text{for all } j \geq 1.$$  

Thus this sequence is bounded in $H_0^1(\Omega; \mathbb{R}^3)$ so it will be compact in $L^2(\Omega; \mathbb{R}^3)$ from Rellich’s theorem. This is impossible if they are $L^2-$orthonormal, so the set must be finite. Thus the second sentence of the theorem holds.

Suppose $B$ is not a maximal C-orthonormal set in $V_0^1(\Omega)$. Then there is a $w \in V_0^1(\Omega)$ with $\|w\|_{C^1} = 1$ and $\langle w, v^{(j)} \rangle_C = 0$ for all $j \geq 1$. Then $\tilde{w} := w/\|w\|$ will be in $B_j$ for every $j \geq 1$. But $C(\tilde{w})$ is finite so this contradicts the definition of $\lambda_j$ for $j$ large enough. Thus the set $B$ must be maximal as claimed. \qed
This theorem shows that a class of Stokes eigenfields defined by the above method of successive optimization generates an orthonormal basis of the space $V_0^1(\Omega)$. The next two sections will describe some consequences of this.

This weak characterization of the eigenfields also yields the following interesting result. A vector field $v \in H^1(\Omega, \mathbb{R}^3)$ is said to be in $H^1_{\nu_0}(\Omega; \mathbb{R}^3)$ provided its normal boundary trace $v \cdot \nu = 0$ on $\partial \Omega$.

**Proposition 3.4.** Assume (B1) holds and $v^{(j)}$ in $V_0^1(\Omega)$ is an eigenfield of (3.1). The $\omega^{(j)}$ is a incompressible field which is in the space $H^1_{\nu_0}(\Omega; \mathbb{R}^3)$.

**Proof.** The proof that $\omega^{(j)}$ is incompressible is straightforward. From (3.1) it will satisfy

$$|\int_{\Omega} \omega^{(j)} \cdot \text{curl} w \, d^3x| \leq \lambda_j \|w\|_2 \quad \text{for all} \ w \in V_0^1(\Omega).$$

This implies that $\text{curl} \ \omega^{(j)} \in L^2(\Omega; \mathbb{R}^3)$. Evaluate

$$\int_{\Omega} \omega^{(j)} \cdot \nabla \varphi \, d^3x = \int_{\partial \Omega} \varphi (\omega^{(j)} \cdot \nu) \, d\sigma = 0$$

using (2.5) and the fact that $v^{(j)} = 0$ on $\partial \Omega$. This holds for all smooth $\varphi$ on $\overline{\Omega}$ so the boundary trace $\omega^{(j)} \cdot \nu \equiv 0$ on $\partial \Omega$. When this holds, and the divergence and curl of $\omega^{(j)}$ are both $L^2$, it follows that $\omega^{(j)} \in H^1_{\nu_0}(\Omega; \mathbb{R}^3)$ from theorem 6.1, chapter 8 of Duvaut-Lions [6].

Let $A$ be a smooth incompressible vector field with compact support in $\Omega$. Substitute $w = \text{curl} \ A$ in (3.1), then use of (2.5) leads to

$$(3.6) \quad \int_{\Omega} [\text{curl} \ \omega^{(j)} \cdot \text{curl} \ A - \lambda_j \omega^{(j)} \cdot A] \, d^3x = 0 \quad \text{for all such} \ A.$$

This implies that $\omega^{(j)}$ is an eigenfield of the zero-flux curl $^2$ eigenproblem on $\Omega$. This problem is of central importance in magnetostatics and some of its properties are described in [1] and [2]. That is, the eigenvalues of the Stokes eigenproblem are a subset of those for the zero flux curl $^2$ eigenproblem. When $\Omega$ is not simply connected, the later problem has 0 as a eigenvalue of nonzero multiplicity so these two problems will not be isospectral in this case.

4. ENERGY, ENSTROPHY AND HELICITY FORMULAE

The representation of an incompressible field with respect to this Stokes basis provides some useful spectral formulae for the energy and enstrophy of the incompressible fields obeying no-slip boundary conditions.

A vector field $v \in V^1_0(\Omega)$ will have a representation of the form

$$(4.1) \quad v = \sum_{j=1}^{\infty} c_j \tilde{v}^{(j)} \quad \text{with} \ c_j := [v, \tilde{v}^{(j)}]_C$$
since $\mathcal{B}$ is an orthonormal basis of $V_0^1(\Omega)$. Note that the coefficients are also given by

\begin{equation}
  c_j = \sqrt{\lambda_j} \int_\Omega v \cdot v^{(j)} \, d^3x
\end{equation}

upon using (3.1) and the definition of the $\tilde{v}^{(j)}$. Moreover, from Parseval’s equality, an expression of the form (4.1) represents a field in $V_0^1(\Omega)$ if and only if

\begin{equation}
  \|\text{curl} \ v\|^2 = \|v\|^2_C = \sum_{j=1}^\infty |c_j|^2 < \infty
\end{equation}

This quantity is often called the \textit{enstrophy} of the vector field $v$.

The \textit{kinetic energy} of the field $v$ will be

\begin{equation}
  \|v\|^2 = \sum_{j=1}^\infty \lambda_j^{-1}|c_j|^2
\end{equation}

upon using (3.1) and the orthogonality relations.

The \textit{helicity} of a field $v$ is the functional

\begin{equation}
  \mathcal{H}(v) := \int_\Omega v \cdot \text{curl} \ v \, d^3x
\end{equation}

and this will be finite for all $v \in V_0^1(\Omega)$. Moreover use of (4.1) shows that this quantity can be expressed in terms of the values of

\[< v^{(j)}, \omega^{(k)} > = < v^{(k)}, \omega^{(j)} > \quad \text{for} \ j, k \geq 1.\]

The following relates the helicity of these basis fields and that of their vorticity.

**Proposition 4.1.** Assume (B1) holds, $v^{(j)}$ is a Stokes eigenfield corresponding to the eigenvalue $\lambda_j$ and $\omega^{(j)} = \text{curl} \ v^{(j)}$. Then $\mathcal{H}(\omega^{(j)}) = \lambda_j \mathcal{H}(v^{(j)})$.

**Proof.** When $v^{(j)}$ is a Stokes eigenfield corresponding to the eigenvalue $\lambda_j$, then Proposition 2.1 and (3.1) implies

\[\text{curl}^2 v^{(j)} = \lambda_j v^{(j)} + \nabla p \quad \text{on} \ \Omega\]

for some $p$. Take scalar products with $\omega^{(j)}$ and integrate over $\Omega$, then the result follows. \qed

5. \textbf{Reconstruction of the Velocity from the Vorticity}

A well-known problem for fluid flows is the problem of determining the velocity of a flow field from the vorticity of a flow in a bounded region. For 2 dimensional flows this is discussed in many texts. An analysis of this for incompressible flows in a bounded three dimensional region may be found in Auchmuty [3].

Suppose $v \in V_0^1(\Omega)$ is given by (4.1), then the coefficients $c_j$ are given by

\begin{equation}
  c_j = \int_\Omega \text{curl} \ v \cdot \tilde{\omega}^{(j)} \, d^3x
\end{equation}
In particular, these coefficients are determined in terms of the vorticity and the vorticities of the Stokes basis fields. That is, given the vorticity of a field, we need only evaluate these integrals to determine $c_j$ and then, from (4.1),

$$v = \sum_{j=1}^{\infty} \lambda_j \langle \text{curl } v, \omega^{(j)} \rangle v^{(j)} \quad \text{on } \Omega.$$  

(5.2)

This is valid for any incompressible field satisfying no-slip boundary conditions and of finite enstrophy.

REFERENCES


