LAPLACIAN EIGENPROBLEMS ON PRODUCT REGIONS AND TENSOR PRODUCTS OF SOBOLEV SPACES.

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Abstract. Characterizations of eigenvalues and eigenfunctions of the Laplacian on a product domain $\Omega_p := \Omega_1 \times \Omega_2$ are obtained. When zero Dirichlet, Robin or Neumann conditions are specified on each factor, then the eigenfunctions on $\Omega_p$ are precisely the products of the eigenfunctions on the sets $\Omega_1, \Omega_2$ separately. There is a related result when Steklov boundary conditions are specified on $\Omega_2$. These results enable the characterization of $H^1(\Omega_p)$ and $H^1_0(\Omega_p)$ as tensor products and descriptions of some orthogonal bases of the spaces. A different characterization of the trace space of $H^1(\Omega_p)$ is found.

1. Introduction

This paper treats some questions related to $H^1$-Sobolev spaces on product regions $\Omega_p := \Omega_1 \times \Omega_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. A particular interest is the representation of such spaces as tensor products of Sobolev spaces on $\Omega_1, \Omega_2$. Such tensor products provide a formalism for the separation of variables methodology used in approximating solutions of partial differential equations. In particular we treat questions concerning when products of Laplacian eigenfunctions are orthogonal bases of these Hilbert-Sobolev spaces.

The construction of orthogonal bases of various $H^1$-Sobolev spaces in terms of eigenfunctions of the Laplacian on a region is standard and justifies many of the approximation methods used in science and engineering. In particular it enables a reduction of dimension that is often very important for effective approximation schemes. A natural question is whether all the Laplacian eigenfunctions on a product region are products of Laplacian eigenfunctions on the individual factors $\Omega_1, \Omega_2$. This is known to hold when zero-Dirichlet boundary conditions are imposed on both factors $\Omega_1, \Omega_2$ as discussed in section 4.

The analysis here treats weak ($H^1$-)solutions of the eigenproblem. This avoids many issues about regularity; some of which arise from the fact that product regions are not $C^1$ at their “corners” and allows us to prove results that hold for many of the situations studied in numerical computations. The case of zero-Neumann boundary conditions is studied in Section 6. Various combinations of eigenproblems with zero Dirichlet, Robin and Neumann boundary conditions on the factors are described and analyzed in Sections 7 - 9.

Date: July 25, 2015.

The authors gratefully acknowledges research support by NSF award DMS 11008754.


Key words and phrases. Laplacian eigenproblems, Tensor Products, Bases of Sobolev spaces, Steklov eigenproblems.
The situation where Steklov boundary conditions are prescribed on $\Omega_2$ and zero Dirichlet, Neumann, or Robin boundary conditions hold on $\Omega_1$ is studied in Section 10. These lead to a two-parameter eigenproblem on $\Omega_p$. Finally in Section 11, a tensor product characterization of the trace space of functions on the boundary of $\Omega_p$ is described. This provides a different orthogonal basis for the trace space of $H^1$ to that described by the first author in [5].

2. Definitions and Notation.

A region $\Omega$ is a non-empty, connected, open subset of $\mathbb{R}^N$. Its closure is denoted $\overline{\Omega}$ and its boundary $\partial \Omega := \overline{\Omega} \setminus \Omega$. A standard assumption about the region is the following.

(B1): $\Omega$ is a bounded region in $\mathbb{R}^N$ and its boundary $\partial \Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.

When this holds there is an outward unit normal $\nu$ defined at $\sigma$ a.e. point of $\partial \Omega$. The definitions and terminology of Evans and Gariepy [9], will be followed except that $\sigma$, $d\sigma$, respectively, will represent Hausdorff $(N - 1)$-dimensional measure and integration with respect to this measure. All functions in this paper will take values in $\mathbb{R} := [-\infty, \infty]$ and derivatives $D_j u$ are taken in the weak sense.

The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial \Omega, d\sigma)$, $1 \leq p \leq \infty$ are defined in the standard manner and have the usual $p$-norms denoted by $\|u\|_p$ and $\|u\|_{p,\partial \Omega}$.

Let $H^1(\Omega)$ and $H^1_0(\Omega)$ be the usual real Sobolev space of functions on $\Omega$. $H^1(\Omega)$ is a real Hilbert space under the standard $H^1$-inner product

$$[u, v]_{1,2} := \int_{\Omega} [u(x) v(x) + \nabla u(x) \cdot \nabla v(x)] \, dx.$$  \hspace{1cm} (2.1)

Here $\nabla u := (D_1 u, \ldots, D_N u)$ is the gradient of the function $u$ and the associated norm may sometimes be denoted $\|u\|_{1,2,\Omega}$ when there could be questions about the region.

This paper will prove various results about Laplacian eigenproblems on product regions $\Omega_p := \Omega_1 \times \Omega_2$ with $\Omega_i \subset \mathbb{R}^{N_i}$. Here points in $\Omega_1$, $\Omega_2$ are denoted by $x = (x_1, x_2, \ldots, x_{N_1})$ and $y = (y_1, y_2, \ldots, y_{N_2})$, respectively, so that points in the product region $\Omega_p$ are denoted by $(x, y)$. It is well known that the eigenfunctions of the Laplacian on a region $\Omega$ may be found that are bases of various Lebesgue and Sobolev spaces of the form $L^2(\Omega)$, $H^1_0(\Omega)$ and $H^1(\Omega)$. When $\Omega$ is a product region $\Omega_p$, then under quite general conditions, it is shown that various classes of eigenfunctions of the Laplacian on $\Omega_p$ are products of the eigenfunctions of the Laplacian on the factors $\Omega_1, \Omega_2$ respectively. A consequence is that the Hilbert-Sobolev spaces $H^1_0(\Omega_p)$, $H^1(\Omega_p)$ are tensor products of the relevant spaces on $\Omega_1, \Omega_2$ respectively.

In this paper we shall use various standard results from the calculus of variations and convex analysis. When terms are not otherwise defined they should be taken as in Attouch, Buttazzo and Michaille [2]. Background material on such methods may be found in Blanchard and Brüning [8] or Zeidler [12]. Some of the notation and many of the methods in this paper are derived from the results developed in the recent paper of Auchmuty [5].

Our interest in studying these Laplacian eigenfunction problems is partially motivated by questions regarding possible orthonormal bases of various Hilbert-Sobolev spaces on product regions Ω := Ω_1 × Ω_2. In particular, we will prove results that generalize the well-known result that \( L^2(\Omega_1) \otimes L^2(\Omega_2) \) with \( \otimes \) denoting the tensor product of the spaces for Lebesgue spaces on \( \Omega_p \).

Here some constructions and results about Hilbert tensor products will be stated for use in the following sections. The description is specially tailored for the current situation; in particular the use of dual spaces is avoided so that distributions are not needed. Let \( H_1 \subset L^2(\Omega_1), H_2 \subset L^2(\Omega_2) \) be two real separable Hilbert spaces with inner products \( \langle \cdot, \cdot \rangle_{H_1} \), and \( \langle \cdot, \cdot \rangle_{H_2} \) respectively. Let \( \mathcal{E}_1 := \{ e_j : j \in J_1 \}, \mathcal{E}_2 := \{ f_k : k \in J_2 \} \) be orthonormal bases of \( H_1, H_2 \) respectively. Without loss of generality all functions are assumed to be Borel measurable and the tensor product of \( e_j \) and \( f_k \) is defined to be the function \( e_j \otimes f_k \) given by

\[
(e_j \otimes f_k)(x, y) := e_j(x) f_k(y) \quad \text{for } (x, y) \in \Omega_p.
\]

Then \( e_j \otimes f_k \) is Borel measurable on \( \Omega_1 \times \Omega_2 \) with \( \| e_j \otimes f_k \|_{L^2(\Omega_p)} = \| e_j \|_{L^2(\Omega_1)} \| f_k \|_{L^2(\Omega_2)} \) from Fubini’s theorem. Functions of this “separated variables”-form will be called dyads.

Let \( \mathcal{E}_\otimes := \{ e_j \otimes f_k : j \in J_1, k \in J_2 \} \) and define \( H_1 \otimes_F H_2 \) to be the vector space of all finite linear combinations of functions in \( \mathcal{E}_\otimes \). Then \( g \in H_1 \otimes_F H_2 \) implies that

\[
g(x, y) = \sum_{j \in J_1, k \in J_2} \hat{g}_{jk} e_j(x) f_k(y).
\]

with only finitely many \( \hat{g}_{jk} \) non-zero. Define an inner product on \( H_1 \otimes_F H_2 \) by

\[
\langle g, h \rangle_\otimes := \sum_{j, k} \hat{g}_{jk} \hat{h}_{jk}.
\]

Then \( \mathcal{E}_\otimes \) will be an orthonormal set in \( H_1 \otimes_F H_2 \) and, from (3.2),

\[
\hat{g}_{jk} = \langle g, e_j \otimes f_k \rangle_\otimes.
\]

The \textit{Hilbert tensor product} \( H_1 \otimes H_2 \) is the completion of \( H_1 \otimes_F H_2 \) with respect to the inner product \( \langle \cdot, \cdot \rangle_\otimes \). It is straightforward to verify that a function \( g \) is in \( H_1 \otimes H_2 \) if and only if it has a representation of the form (3.2) with \( \sum_{j,k} \hat{g}_{jk}^2 < \infty \). Thus \( \mathcal{E}_\otimes \) is an orthonormal basis of \( H_1 \otimes H_2 \) and the Parseval equality implies that (3.3) holds for all \( g, h \in H_1 \otimes H_2 \) with Fourier coefficients defined by (3.4). Moreover, when \( u \in H_1, v \in H_2 \), then \( u \otimes v \in H_1 \otimes H_2 \) with Fourier coefficients \( \hat{u} \hat{v}_k = \langle u, e_j \rangle_{H_1} \langle v, f_k \rangle_{H_2} \) and

\[
\| u \otimes v \|_\otimes = \| u \|_{H_1} \| v \|_{H_2}.
\]

Since two Hilbert spaces with orthonormal bases that are in 1-1 correspondence are linearly isomorphic the following holds.

**Theorem 3.1.** Let \( H_1, H_2 \) be real separable Hilbert spaces as above with orthonormal bases \( \mathcal{E}_1, \mathcal{E}_2 \) and \( H \) is a real Hilbert space with basis \( \mathcal{E}_\otimes \). Then \( H \) is linearly isomorphic to \( H_1 \otimes H_2 \).
A simple result about tensor products that will be used later is the following. Its proof is standard analysis.

**Theorem 3.2.** Suppose that $H_1 = V_1 \oplus W_1$ and $H_2 = V_2 \oplus W_2$ are orthogonal decompositions of $H_1, H_2$, respectively. Then

$$H_1 \otimes H_2 = (V_1 \otimes V_2) \oplus (V_1 \otimes W_2) \oplus (W_1 \otimes V_2) \oplus (W_1 \otimes W_2).$$

(3.6)

Moreover, the tensor product of corresponding orthonormal bases of $V_1, V_2$ is an orthonormal basis for $V_1 \otimes V_2$, and a similar result holds for the remaining three terms in the orthogonal decomposition of $H_1 \otimes H_2$.

4. **Dirichlet Laplacian Eigenproblems on $\Omega$ and $\Omega_p$.**

The *Dirichlet Laplacian eigenproblem* on a region $\Omega \subset \mathbb{R}^N$ is the problem of finding nontrivial solutions $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ of the equation

$$\int_{\Omega} \nabla u \cdot \nabla h \, dx = \lambda \int_{\Omega} u \, h \, dx \quad \text{for all } h \in H_0^1(\Omega).$$

(4.1)

This is the weak form of the eigenvalue problem of finding nontrivial solutions of the system

$$-\Delta u(x) = \lambda u(x) \quad \text{on } \Omega \quad \text{with } u(x) = 0 \quad \text{on } \partial \Omega.$$  

(4.2)

Here $\Delta$ is the Laplacian on the region $\Omega$.

The classical result about these problems may be stated as follows (see, for instance, Theorem 8.3.2 and Proposition 8.3.1 of Attouch et al. [2]). Problem (4.1) has a sequence $\Lambda := \{\lambda_j : j \in \mathbb{N}\}$ of strictly positive eigenvalues, repeated according to multiplicity and $\lambda_j \to \infty$, and an associated sequence of eigenfunctions $\mathcal{E} := \{e_j : j \in \mathbb{N}\}$. The sequence $\mathcal{E}$ forms an orthonormal basis of $L^2(\Omega)$, and the sequence $\tilde{\mathcal{E}} := \{\lambda_j^{-1/2} e_j : j \in \mathbb{N}\}$ forms an orthonormal basis for $H_0^1(\Omega)$ with respect to the inner product $\langle u, v \rangle_{\nabla} := \int_{\Omega} \nabla u \cdot \nabla v \, dx$, so that

$$\int_{\Omega} \nabla e_j \cdot \nabla e_j \, dx = \lambda_j.$$  

(4.3)

When $\Omega_p := \Omega_1 \times \Omega_2$ is a product region, then the individual Dirichlet Laplacian eigenproblems on $\Omega_1, \Omega_2$ are those of finding nontrivial solutions of equation (4.1) with $\Omega_1, \Omega_2$ in place of $\Omega$.

When $\Omega = \Omega_1$, the sequence of eigenvalues will be denoted $\Lambda_1 := \{\lambda_{1j} : j \in \mathbb{N}\}$ and the eigenfunctions $\mathcal{E}_1 = \{e_{1j} : j \in \mathbb{N}\}$; $\mathcal{E}_1$ is an orthonormal basis of $L^2(\Omega_1)$. When $\Omega = \Omega_2$, the sequence of eigenvalues will be denoted $\Lambda_2 := \{\lambda_{2k} : k \in \mathbb{N}\}$ and the eigenfunctions $\mathcal{E}_2 := \{e_{2k} : k \in \mathbb{N}\}$ with $\mathcal{E}_2$ an orthonormal basis of $L^2(\Omega_2)$.

Also, the sequences $\tilde{\mathcal{E}}_1 := \{\lambda_{1j}^{-1/2} e_{1j} : j \in \mathbb{N}\}$ and $\tilde{\mathcal{E}}_2 := \{\lambda_{2k}^{-1/2} e_{2k} : k \in \mathbb{N}\}$ will be orthonormal bases of $H_0^1(\Omega_1), H_0^1(\Omega_2)$ with respect to the inner products

$$\langle u, v \rangle_{\nabla, \Omega_1} := \int_{\Omega_1} \nabla_x u \cdot \nabla_x v \, dx \quad \text{and} \quad \langle u, v \rangle_{\nabla, \Omega_2} := \int_{\Omega_2} \nabla_y u \cdot \nabla_y v \, dy$$  

(4.4)

where $\nabla_x, \nabla_y$ are the corresponding gradient operators on the regions $\Omega_1, \Omega_2$. 
The following result about product Dirichlet eigenfunctions is a weak version of Theorem 2 in Chapter 11 of Strauss [11]. It is worth noting that there are regularity issues for eigenfunctions of the Laplacian on product domains as the boundary of $\Omega_p$ is not a union of $C^1$-manifolds but instead has “corners”. So most results here are confined to weak solutions of the eigenproblems.

**Theorem 4.1.** Assume $\Omega_1, \Omega_2$ satisfy (B1), and let $\mathcal{E}_1, \mathcal{E}_2$ be the sequences of Dirichlet Laplacian eigenfunctions on $\Omega_1, \Omega_2$ described above. Then $u_{jk} := e_{1j} \otimes e_{2k}$ is a Dirichlet-Laplacian eigenfunction for the problem (4.1) on the product region $\Omega_p = \Omega_1 \times \Omega_2$ corresponding to the eigenvalue $\lambda_{1j} + \lambda_{2k}$.

**Proof.** Let $h \in C_c^1(\Omega_p)$. Then $h(x, \cdot) \in C_c^1(\Omega_2)$ and $h(\cdot, y) \in C_c^1(\Omega_1)$ for any $(x, y) \in \Omega_p$. Substituting $u = u_{jk}$ in the left side of (4.1) yields

$$
\int_{\Omega_2} \int_{\Omega_1} [e_{2k}(y)(\nabla_x e_{1j}(x) \cdot \nabla_x h(x, y)) + e_{1j}(x)(\nabla_y e_{2k}(y) \cdot \nabla_y h(x, y))] \, dx \, dy
$$

$$
= \int_{\Omega_2} e_{2k}(y) \int_{\Omega_1} \nabla_x e_{1j}(x) \cdot \nabla_x h(x, y) \, dx \, dy + \int_{\Omega_1} e_{1j}(x) \int_{\Omega_2} \nabla_y e_{2k}(y) \cdot \nabla_y h(x, y) \, dx \, dy
$$

from Fubini's theorem. This equals

$$
\lambda_{1j} \int_{\Omega_2} e_{2k}(y) \int_{\Omega_1} e_{1j}(x) h(x, y) \, dx \, dy + \lambda_{2k} \int_{\Omega_1} e_{1j}(x) \int_{\Omega_2} e_{2k}(y) h(x, y) \, dy \, dx
$$

upon using the eigenequations on $\Omega_1, \Omega_2$. This is the right side of (4.1) with $\lambda_{1j} + \lambda_{2k}$ and $u_{jk}$ in place of $\lambda$ and $u$, respectively. A density argument then yields that $\lambda_{1j} + \lambda_{2k}$ is an eigenvalue of this problem and $u_{jk}$ is an associated eigenfunction. \hfill \Box

Let $\mathcal{U} := \{u_{jk} : j, k \in \mathbb{N}\}$. One observes that this set is orthonormal in $L^2(\Omega_p)$. For each $j, k$ define $\bar{u}_{jk} := (\lambda_{1j} + \lambda_{2k})^{-1/2}u_{jk}$ and let $\mathcal{U}$ be the family of all such functions; this set is orthonormal in $H^1_0(\Omega_p)$.

**Theorem 4.2.** Assume $\Omega_1, \Omega_2$ satisfy (B1), and let $\mathcal{U}, \mathcal{U}$ be the sequences of eigenfunctions on $\Omega_p$ as described above. Then $\mathcal{U}$ is an orthonormal basis of $L^2(\Omega_p)$ and $\mathcal{U}$ is an orthonormal basis of $H^1_0(\Omega_p)$.

**Proof.** From the observations in the preceding paragraph these sets are orthonormal. Here we show that the sets are maximal. Suppose there is a nonzero $h \in L^2(\Omega_p)$ such that

$$
\int_{\Omega_p} u_{jk}(x, y) h(x, y) \, dx \, dy = 0 \quad \text{for all } j, k \in \mathbb{N}.
$$

Fubini’s theorem then implies

$$
\int_{\Omega_1} e_{1j}(x) \int_{\Omega_2} e_{2k}(y) h(x, y) \, dy \, dx = 0 \quad \text{for all } j \in \mathbb{N},
$$

Since $\mathcal{E}_1$ is an orthonormal basis of $L^2(\Omega_1)$, this implies $\int_{\Omega_1} e_{2k}(y) h(x, y) \, dy = 0$ for a.e. $x \in \Omega_1$. Since this holds for all $k \in \mathbb{N}$, $h(x, y) = 0$ for a.e. $(x, y) \in \Omega_p$ as $\mathcal{E}_2$ is an orthonormal basis of $L^2(\Omega_2)$. This contradicts the assumption that $h$ is non-zero so $\mathcal{U}$ is a maximal set in $L^2(\Omega_p)$. 

Suppose that $\lambda$ is an eigenvalue of this problem but $\lambda$ not equal to any of the $\lambda_{1j} + \lambda_{2k}$. The associated eigenfunction $e$ satisfies (4.1) for then is orthogonal to each $u_{jk}$ both in $H^1_0(\Omega_p)$ and $L^2(\Omega_p)$. The first part of the proof shows this is not possible. If, on the other hand, $e$ satisfies (4.1) for $\lambda$ equal to a fixed $\lambda_{1j} + \lambda_{2k}$, then the function

$$\psi(x,y) := e(x,y) - \sum_{\lambda_{1j} + \lambda_{2k} = \lambda} c_{jk} u_{jk} \quad \text{with} \quad c_{jk} := \frac{\int_{\Omega_p} u_{jk} \, dx \, dy}{\int_{\Omega_p} u_{jk}^2 \, dx \, dy}$$

would be orthogonal to each $u_{jk}$ both in $H^1_0(\Omega_p)$ and $L^2(\Omega_p)$. The first part of the proof, in this case, shows $\psi$ is zero a.e. on $\Omega_p$, or that $e$ belongs to the subspace associated to $\lambda = \lambda_{1j} + \lambda_{2k}$. Hence, the only Dirichlet Laplacian eigenfunctions on $\Omega_p$ are the dyads $u_{jk} = e_{1j} \otimes e_{2k}$.

Finally, if $h$ is orthogonal to each $\tilde{u}_{jk}$ in $H^1_0(\Omega_p)$, then from equation (4.1) it must be $L^2$-orthogonal to each $\tilde{u}_{jk}$ so from the first part of this proof we have a contradiction. Thus $\tilde{U}$ is maximal in $H^1_0(\Omega_p)$. \hfill \Box

**Corollary 4.3.** Assume $\Omega_1, \Omega_2$ satisfy (B2), and let $U$ be as described above. Then the only eigenvalues of problem (4.1) on the region $\Omega_p = \Omega_1 \times \Omega_2$ are $\lambda_{1j} + \lambda_{2k}$ with $j,k \in \mathbb{N}$, and $H^1_0(\Omega_p) = H^1_0(\Omega_1) \otimes H^1_0(\Omega_2)$.

**Proof.** This follows from Theorem 3.1 and the above result. \hfill \Box

## 5. Equivalent Inner Products and Boundary Regularity

To extend the preceding results to Robin, Neumann and Steklov eigenproblems some further conditions must be imposed on the regions $\Omega_1, \Omega_2$. These conditions are essentially conditions on the regularity of the boundaries of the region that yield compact imbedding results for $H^1(\Omega)$. For simplicity the following definitions are given in terms of a generic region $\Omega \subset \mathbb{R}^N$ satisfying (B1).

The region $\Omega$ is said to satisfy *Rellich’s theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_*$ where $p_* := 2N/(N-2)$ when $N \geq 3$, or $p_* = \infty$ when $N = 2$. There are a number of different criteria on $\Omega$ and $\partial \Omega$ that imply this result. When (B1) holds it is Theorem 1 in Section 4.6 of [9]; see also Amick [1].

When (B1) holds and $u \in W^{1,1}(\Omega)$ then the trace of $u$ on $\partial \Omega$ is well-defined and is a Lebesgue integrable function with respect to $\sigma$; see [9], Section 4.2 for details. The trace map $\gamma$ is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial \Omega$. In surface integrals, we will often use $u$ in place of $\gamma(u)$ when considering the trace of a function on $\partial \Omega$. The region $\Omega$ is said to satisfy the *$L^2$-compact trace theorem* provided the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial \Omega, d\sigma)$ is compact. The regions for our problems will generally be required to satisfy

**(B2):** $\Omega$ and $\partial \Omega$ satisfy (B1), the Rellich theorem and the $L^2$-compact trace theorem.
Many of the following results are based on choosing appropriate inner products on $H^1(\Omega)$. These will include the $b$-inner product on $H^1(\Omega)$ given by

$$\langle u, v \rangle_b := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} b \, u v \, d\sigma, \quad (5.1)$$

The corresponding norm will be denoted $\|u\|_b$. When there are different regions involved in the same problem these inner products and norms are written as $\langle u, v \rangle_{b, \Omega}$ and $\|u\|_{b, \Omega}$, accordingly.

The boundary integral in (5.1) has a weight function $b : \partial \Omega \to (0, \infty)$ that satisfies

(B3): $b$ is Borel measurable, $b \in L^\infty(\partial \Omega, d\sigma)$ with $b \geq b_0 > 0$ $\sigma$-a.e. on $\partial \Omega$.

The associated boundary quadratic form is denoted $B(u) := \int_{\partial \Omega} b u^2 \, d\sigma$, and it is shown in Theorem 3.1 of [3] that $B$ is weakly continuous on $H^1(\Omega)$.

The following result about these bilinear forms is crucial for many later results.

**Theorem 5.1.** Assume (B1)-(B3) hold and the bilinear form $\langle \cdot, \cdot \rangle_b$ is given by (5.1). Then this $b$-inner product is equivalent to the standard inner product on $H^1(\Omega)$ given by (2.1).

**Proof.** The proof of Theorem 7.2 of [5] shows (5.1) defines an inner product equivalent to the standard $H^1$-inner product.

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6. **Neumann Laplacian Eigenproblems on $\Omega$ and $\Omega_p$.**

The *Neumann Laplacian eigenproblem* on a region $\Omega \subset \mathbb{R}^N$ is the problem of finding nontrivial solutions $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ of the system

$$\int_{\Omega} \nabla u \cdot \nabla h \, dx = \lambda \int_{\Omega} u h \, dx \quad \text{for all } h \in H^1(\Omega). \quad (6.1)$$

This is the weak form of the eigenvalue problem of finding nontrivial solutions of the system

$$-\Delta u(x) = \lambda u(x) \quad \text{on } \Omega \quad \text{with } D_\nu u = 0 \quad \text{on } \partial \Omega \quad (6.2)$$

with $D_\nu u := \nabla u \cdot \nu$.

A first result about these Neumann Laplacian eigenproblems on bounded regions is well-known subject to various different assumptions about the region $\Omega$. Here a statement that can be proved using the analysis of Section 4 of [5] will be used. Take $V = H^1(\Omega)$, $H = L^2(\Omega)$ in that paper and let

$$a(u, v) := [u, v]_{1,2} \quad \text{and} \quad m(u, v) := \langle u, v \rangle_2 \quad (6.3)$$

where $[\cdot, \cdot]_{1,2}$ and $\langle \cdot, \cdot \rangle_2$ are the standard $H^1$- and $L^2$-inner products, respectively.

Obviously $\lambda = 0$ is an eigenvalue of (6.1) with the associated eigenfunction $f(x) \equiv 1$. To obtain the sequence of all Neumann Laplacian eigenvalues, consider the sequence of constrained variational problems $P_k$ as described by (4.1) in [5]. This generates a sequence $\mathcal{F} := \{f_k : k \in \mathbb{N}\}$ of $H^1$-orthonormal functions satisfying

$$[f_k, h]_{1,2} = \tilde{\lambda}_k \langle f_k, h \rangle_2 \quad \text{for all } h \in H^1(\Omega) \quad (6.4)$$

where $\lambda_k$ and $\tilde{\lambda}_k$ are the corresponding eigenvalues.
with the values $\tilde{\lambda}_k$ being strictly positive.

Define $\lambda_k := \tilde{\lambda}_k - 1$ and let $\Lambda := \{\lambda_k : k \in \mathbb{N}\}$. Then $f_k$ satisfies (6.1) with Neumann Laplacian eigenvalue $\lambda = \lambda_k$, and consequently

$$\langle f_j, f_k \rangle_2 = \delta_{jk}/(1 + \lambda_k) \quad \text{for } j, k \in \mathbb{N}. \quad (6.5)$$

Take $\tilde{F} := \{(1 + \lambda_k)^{1/2} f_k : k \in \mathbb{N}\}$. These eigendata satisfy the following result.

**Theorem 6.1.** Assume (B1), (B2) hold and $\Lambda$, $F$, $\tilde{F}$ are defined as above. Then $\Lambda$ is an increasing sequence with $\lambda_k \to \infty$, $F$ is a maximal orthonormal set in $H^1(\Omega)$ and $\tilde{F}$ is a maximal orthonormal set in $L^2(\Omega)$.

**Proof.** The bilinear form $a$ in (6.3) automatically obeys condition (A1) of [5]. The form $m$ obeys conditions (A2) and (A4) of [5] from our assumption (B2). Thus the first parts of this theorem follow since the $\tilde{\lambda}_k$ obey the results from Theorem 4.3 of [5].

From Rellich’s theorem the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact and also it has dense range, so condition (A5) holds and Theorem 4.6 of [5] yields that $\tilde{F}$ is a maximal orthonormal set in $L^2(\Omega)$.

Let $(\Lambda_1, F_1)$ and $(\Lambda_2, F_2)$ be sequences of Neumann Laplacian eigenvalues and eigenfunctions on $\Omega_1, \Omega_2$ generated as above with $\Lambda_1 := \{\lambda_{1k} : k \in \mathbb{N}\}, F_1 := \{f_{1k} : k \in \mathbb{N}\}$ and similarly for $\Lambda_2, F_2$. Consider the family $U$ consisting of the dyads

$$u_{jk} := f_{1j} \otimes f_{2k} \quad \text{with } j, k \in \mathbb{N}. \quad (6.6)$$

These dyads have the following properties.

**Theorem 6.2.** Assume $\Omega_1, \Omega_2$ are regions that satisfy (B1) and (B2), $\Omega_p := \Omega_1 \times \Omega_2$ and $\Lambda_1, \Lambda_2, F_1, F_2, U$ as above. Then each $u_{jk} \in U$ is a Neumann Laplacian eigenfunction of (6.1) on $\Omega_p$ corresponding to the eigenvalue $\lambda_{1j} + \lambda_{2k}$. Moreover, $U$ is orthogonal in $L^2(\Omega_p)$ and also in $H^1(\Omega_p)$.

**Proof.** When $h \in H^1(\Omega_p)$, then $h(x, \cdot) \in H^1(\Omega_2)$ for almost all $x \in \Omega_1$ and $h(\cdot, y) \in H^1(\Omega_1)$ for almost all $y \in \Omega_2$. From the Neumann eigenequation on $\Omega_1$,

$$\int_{\Omega_1} \nabla_x u_{jk}(x, y) \cdot \nabla_x h(x, y) \, dx = \lambda_{1j} f_{2k}(y) \int_{\Omega_1} f_{1j}(x) h(x, y) \, dx$$

for almost all $y \in \Omega_2$. Integrating this over $\Omega_2$, yields

$$\int_{\Omega_2} \int_{\Omega_1} f_{2k}(y) \nabla_x f_{1j}(x) \cdot \nabla_x h(x, y) \, dx \, dy = \lambda_{1j} \int_{\Omega_2} f_{2k}(y) \int_{\Omega_1} f_{1j}(x) h(x, y) \, dx \, dy.$$  

Similarly one has that

$$\int_{\Omega_1} \int_{\Omega_2} f_{1j}(x) \nabla_y f_{2k}(y) \cdot \nabla_y h(x, y) \, dy \, dx = \lambda_{2k} \int_{\Omega_1} f_{1j}(x) \int_{\Omega_2} f_{2k}(y) h(x, y) \, dy \, dx.$$  

Adding these shows that $u_{jk}$ is a solution of (6.1) with eigenvalue $\lambda_{1j} + \lambda_{2k}$ as claimed.

The fact that the $u_{jk}$ are $L^2$-orthogonal follows by using Fubini’s theorem and the fact that the families $F_1, F_2$ are $L^2$-orthogonal on $L^2(\Omega_1), L^2(\Omega_2)$ respectively. This and the eigenequation (6.1) yields the $H^1$-orthogonality of the $u_{jk}$. \qed
From definition, the \( u_{jk} \) satisfy
\[
\langle u_{jk}, u_{jk} \rangle_{2, \Omega_p} = \int_{\Omega_1} f_{1j}(x)^2 \, dx \int_{\Omega_2} f_{2k}(y)^2 \, dy = \frac{1}{(1 + \lambda_{1j})(1 + \lambda_{2k})}.
\] (6.7)
Thus, adding \( \langle u_{jk}, u_{jk} \rangle_{2, \Omega_p} \) to both sides of (6.1) gives
\[
\frac{1}{2} u_{jk} \in \Omega_p
\]

Define \( v_{jk}, \tilde{v}_{jk} \) to be the functions
\[
v_{jk} := (1 + \lambda_{1j})^{1/2}(1 + \lambda_{2k})^{1/2} u_{jk} \quad \text{and} \quad \tilde{v}_{jk} := \frac{(1 + \lambda_{1j})(1 + \lambda_{2k})}{1 + \lambda_{1j} + \lambda_{2k}} u_{jk}
\] (6.8)
and let \( \mathcal{V} := \{ v_{jk} : j, k \in \mathbb{N} \} \) and \( \mathcal{V} := \{ \tilde{v}_{jk} : j, k \in \mathbb{N} \}. \) Then \( \mathcal{V} \) is an orthonormal set in \( H^1(\Omega_p) \), \( \mathcal{V} \) is an orthonormal set in \( L^2(\Omega_p) \) and the following holds.

**Corollary 6.3.** Assume \( \Omega_1, \Omega_2 \) are regions that satisfy (B1) and (B2) and \( \Omega_p := \Omega_1 \times \Omega_2. \) Then \( \mathcal{V} \) is an orthonormal basis of \( H^1(\Omega_p) \), \( \mathcal{V} \) is an orthonormal basis of \( L^2(\Omega_p) \), and \( H^1(\Omega_p) = H^1(\Omega_1) \otimes H^1(\Omega_2). \)

**Proof.** Suppose \( h \in H^1(\Omega_p) \), then from the eigenequation (6.1) and the preceding theorem,
\[
\left[ h, u_{jk} \right]_{2, \Omega_p} = (1 + \lambda_{1j} + \lambda_{2k}) \langle h, u_{jk} \rangle_{2, \Omega_p} \quad \text{for all} \quad j, k \in \mathbb{N}.
\]
From (6.8) each value \( 1 + \lambda_{1j} + \lambda_{2k} \) is strictly positive. If \( \mathcal{V} \) is not a maximal orthogonal set in \( H^1(\Omega_p) \), then there is a nonzero \( \hat{h} \in H^1(\Omega_p) \) with \( \left[ \hat{h}, u_{jk} \right]_{2, \Omega_p} = 0 \) for all \( j, k \). From the first line, \( \langle \hat{h}, u_{jk} \rangle_{2, \Omega_p} = 0 \). However \( \mathcal{V} \) is a maximal orthogonal set in \( L^2(\Omega_p) \) since \( F_1, F_2 \) are maximal orthogonal sets in \( L^2(\Omega_1), L^2(\Omega_p) \), respectively and \( L^2(\Omega_p) = L^2(\Omega_1) \otimes L^2(\Omega_2) \). Thus \( \hat{v} \equiv 0 \) in \( L^2(\Omega_p) \) and a fortiori in \( H^1(\Omega_p) \). Hence \( \mathcal{V} \) is a maximal \( H^1 \)-orthonormal set as claimed. Since \( \mathcal{V} \) is an \( H^1 \)-orthonormal basis of the Hilbert space \( H^1(\Omega_p) \) consisting of dyads of the form \( f_{1j} \otimes f_{2k} \), then \( H^1(\Omega_p) = H^1(\Omega_1) \otimes H^1(\Omega_2) \) from Theorem 3.1.

---

### 7. Robin Laplacian Eigenproblems on \( \Omega \) and \( \Omega_p. \)

The **Robin Laplacian eigenproblem** on a region \( \Omega \subset \mathbb{R}^N \) is the problem of finding nontrivial solutions \( (\lambda, u) \in \mathbb{R} \times H^1(\Omega) \) of the system
\[
\int_{\Omega} \nabla u \cdot \nabla h \, dx + \int_{\partial \Omega} b u h \, d\sigma = \lambda \int_{\Omega} u h \, dx \quad \text{for all} \quad h \in H^1(\Omega).
\] (7.1)
This is the weak form of the eigenvalue problem of finding nontrivial solutions of the system
\[
-\Delta u(x) = \lambda u(x) \quad \text{on} \quad \Omega \quad \text{with} \quad D_{\nu} u + b u = 0 \quad \text{on} \quad \partial \Omega.
\] (7.2)

A general theorem about the spectrum of second order Robin elliptic eigenproblems on bounded regions is given as Theorem 7.2 of [5] and is based on a similar analysis to that of Theorem 6.1 above. Let \( \Lambda := \{ \lambda_k : k \in \mathbb{N} \} \) be the sequence of eigenvalues of (7.1).
repeated according to multiplicity and \( \mathcal{G} := \{ g_k : k \in \mathbb{N} \} \) be an associated sequence of Robin eigenfunctions normalized so that
\[
[g_j, g_k]_b := \int_{\Omega} \nabla g_j \cdot \nabla g_k \, dx + \int_{\partial \Omega} b g_j g_k \, d\sigma = \delta_{jk} \quad \text{for all } j, k \in \mathbb{N}.
\]  
(7.3)
Then the eigenvalue (7.1) implies that
\[
\langle g_j, g_k \rangle_2 = \frac{\delta_{jk}}{\lambda_k} \quad \text{for } j, k \geq 1.
\]  
(7.4)
Define \( \tilde{\mathcal{G}} := \{ \lambda_k^{-1/2} g_k : k \in \mathbb{N} \} \). Then Theorem 7.2 of [5] may then be stated as

**Theorem 7.1.** Assume that (B1)-(B3) hold and \( \Lambda, \mathcal{G}, \tilde{\mathcal{G}} \) are defined as above. Then \( \Lambda \) is an increasing sequence with \( \lambda_k \to \infty \), \( \mathcal{G} \) is a maximal b-orthonormal set in \( H^1(\Omega) \) and \( \tilde{\mathcal{G}} \) is a maximal orthonormal set in \( L^2(\Omega) \).

To prove results about the Robin Laplacian eigenproblem on the product region \( \Omega_p := \Omega_1 \times \Omega_2 \), first note that its boundary \( \partial \Omega_p \) is given by
\[
\partial \Omega_p = (\partial \Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial \Omega_2).
\]
Also, \( \nu(x, y) = (\nu_1(x), 0) \) for \( (x, y) \in \partial \Omega_1 \times \Omega_2 \) and similarly \( \nu(x, y) = (0, \nu_2(y)) \) for \( (x, y) \in \Omega_1 \times \partial \Omega_2 \). Here \( \nu_1, \nu_2 \) are unit normal vectors in \( \mathbb{R}^{N_1}, \mathbb{R}^{N_2} \) respectively. The region \( \Omega_p \) in general has “corners” with no unit outward normal at points \( (x, y) \in \partial \Omega_1 \times \partial \Omega_2 \).

The previous condition (B3) must be modified for Robin problems on \( \Omega_p \). We now require that
\[
\text{(B4): } b : \partial \Omega_p \to [0, \infty] \text{ is such that } b(x, y) = b_1(x) \text{ for a.e. } y \in \Omega_2, \text{ and } b(x, y) = b_2(y) \text{ for a.e. } x \in \Omega_1 \text{ with } b_1, b_2 \text{ satisfying (B3) on } \partial \Omega_1, \partial \Omega_2 \text{ respectively.}
\]
Let \( (\Lambda_1, \mathcal{G}_1) \) and \( (\Lambda_2, \mathcal{G}_2) \) be the Robin Laplacian eigendata on \( \Omega_1, \Omega_2 \) generated as above with \( \Lambda_1 := \{ \lambda_{1j} : j \in \mathbb{N} \}, \mathcal{G}_1 := \{ g_{1j} : j \in \mathbb{N} \} \) and similarly for \( \Lambda_2, \mathcal{G}_2 \). Consider the family \( \mathcal{U} \) consisting of the dyads
\[
u_{jk} := g_{1j} \otimes g_{2k} \quad \text{with } j, k \in \mathbb{N}.
\]  
(7.5)
These functions have the following properties.

**Theorem 7.2.** Assume that (B1)-(B4) hold, \( \Omega_p := \Omega_1 \times \Omega_2 \) and \( \Lambda_1, \Lambda_2, \mathcal{G}_1, \mathcal{G}_2, \mathcal{U} \) as above. Then each \( u_{jk} \) is a Robin Laplacian eigenfunction of (7.1) on \( \Omega_p \) corresponding to the eigenvalue \( \lambda_{1j} + \lambda_{2k} \). Moreover \( \mathcal{U} \) is an orthogonal family in \( H^1(\Omega_p) \) with respect to the b-inner product and is an orthogonal family in \( L^2(\Omega_p) \).

**Proof.** When \( g_{1j}, g_{2k} \) are as above then they are solutions, respectively, of
\[
\int_{\Omega_1} \nabla x g_{1j} \cdot \nabla x h_1 \, dx + \int_{\partial \Omega_1} b_1 g_{1j} h_1 \, d\sigma_1 = \lambda_{1j} \int_{\Omega_1} g_{1j} h_1 \, dx \quad \text{ (7.6)}
\]
\[
\int_{\Omega_2} \nabla y g_{2k} \cdot \nabla y h_2 \, dy + \int_{\partial \Omega_2} b_2 g_{2k} h_2 \, d\sigma_2 = \lambda_{2k} \int_{\Omega_2} g_{2k} h_2 \, dy \quad \text{ (7.7)}
\]
for all \( h_1 \in H^1(\Omega_1) \) and \( h_2 \in H^1(\Omega_2) \). Multiply the first equation by \( g_{2k}(y) h_2(y) \) and integrate over \( \Omega_2 \), and the second equation by \( g_{1j}(x) h_1(x) \) and integrate over \( \Omega_1 \). Add these two equations and observe that the resulting left hand side is \([u_{jk}, h]_{b,\Omega_p}\) where \( h(x, y) = \)}
Let $h_1 = g_{1j}, h_2 = g_{2k}$ in the above formulae to obtain $[g_{1j}, g_{1j}]_{b,\Omega_1} = [g_{2k}, g_{2k}]_{b,\Omega_2} = 1$ and

$$1 = \lambda_{1j} \langle g_{1j}, g_{1j} \rangle_{2,\Omega_1} \quad \text{and} \quad 1 = \lambda_{2k} \langle g_{2k}, g_{2k} \rangle_{2,\Omega_2}.$$  

Multiply the first of these equalities by $\tilde{g}_{jk}$ and integrate over $\Omega_2$ and the second by $g_{1j}$ and integrate over $\Omega_1$. Then

$$\frac{1}{\lambda_{2k}} + \frac{1}{\lambda_{1j}} = (\lambda_{1j} + \lambda_{2k}) \int_{\Omega_\mu} u_{jk}(x,y)^2 \, dx \, dy = [u_{jk}, u_{jk}]_{b,\Omega_\mu} \quad (7.8)$$

Choosing $h_1 = g_{1j}$ with $j \neq k$ or $h_2 = g_{2k}$ with $k \neq j$ in these formulae shows that the functions $u_{jk}$ are orthogonal with respect to the $b$-inner product or the $L^2$–inner product on $\Omega_\mu$. \hfill \Box

Define $w_{jk}, \tilde{w}_{jk}$ to be the functions

$$w_{jk} := \left( \frac{\lambda_{1j} \lambda_{2k}}{\lambda_{1j} + \lambda_{2k}} \right)^{1/2} u_{jk} \quad \text{and} \quad \tilde{w}_{jk} := (\lambda_{1j} \lambda_{2k})^{1/2} u_{jk} \quad (7.9)$$

and let $\mathcal{W} := \{w_{jk} : j, k \in \mathbb{N}\}$ and $\tilde{\mathcal{W}} := \{\tilde{w}_{jk} : j, k \in \mathbb{N}\}$. Then $\mathcal{W}$ is a $b$-orthonormal set in $H^1(\Omega_\mu)$ and $\tilde{\mathcal{W}}$ is orthonormal in $L^2(\Omega_\mu)$.

**Corollary 7.3.** Assume (B1)-(B4) hold, $\Omega_\mu := \Omega_1 \times \Omega_2$ and $\mathcal{W}, \tilde{\mathcal{W}}$ as above. Then $\mathcal{W}$ is a maximal $b$-orthonormal set in $H^1(\Omega_\mu)$ and $\tilde{\mathcal{W}}$ a maximal orthonormal set in $L^2(\Omega_\mu)$.

**Proof.** When $h \in H^1(\Omega_\mu)$, then eigenequation (7.1) and the preceding theorem give

$$[h, u_{jk}]_{b,\Omega_\mu} = (\lambda_{1j} + \lambda_{2k}) \langle h, u_{jk} \rangle_{2,\Omega_\mu} \quad \text{for all} \quad j, k \in \mathbb{N}.$$  

If $\mathcal{W}$ is not a maximal $b$-orthonormal set in $H^1(\Omega_\mu)$, then there is a nonzero $\hat{h} \in H^1(\Omega_\mu)$ with $[\hat{h}, u_{jk}]_{b,\Omega_\mu} = 0$ for all $j, k \in \mathbb{N}$. Thus $\langle \hat{h}, u_{jk} \rangle_{2,\Omega_\mu} = 0$ for all $j, k$ as the individual Robin eigenvalues always are strictly positive. However $\mathcal{G}_1, \mathcal{G}_2$ are maximal orthogonal sets in $L^2(\Omega_1), L^2(\Omega_2)$ respectively so their tensor products are a maximal orthogonal set in $L^2(\Omega_\mu)$. Hence $\hat{h} = 0$ in $L^2(\Omega_\mu)$ and thus also in $H^1(\Omega_\mu)$. Therefore $\mathcal{W}$ is a maximal orthogonal set in $H^1(\Omega_\mu)$. \hfill \Box

8. **Dirichlet-Neumann Laplacian Eigenproblems on $\Omega_\mu$.**

In the preceding sections, the eigenproblems on the regions $\Omega_1, \Omega_2$ were of the same type. In this section the Laplacian eigenproblem on $\Omega_\mu$ with Dirichlet conditions on $\Omega_1$ and Neumann conditions on $\Omega_2$ is considered.

This problem can be posed in a number of different ways. The simplest appears to be to seek eigenfunctions in the tensor product space $V := H^1_0(\Omega_1) \otimes H^1(\Omega_2)$. 


It is straightforward to verify that
\[ [u, v]_{\nabla, \Omega_p} := \int_{\Omega_p} \left[ \nabla_x u \cdot \nabla_x v + \nabla_y u \cdot \nabla_y v \right] dx dy \quad (8.1) \]
is an inner product to this space \( V \).

The Dirichlet Neumann Laplacian (DNL) eigenproblem on the product region \( \Omega_p \) is the problem of finding nontrivial solutions \((\lambda, u) \in \mathbb{R} \times V\) of the system
\[
\int_{\Omega_p} \nabla u \cdot \nabla h \ dx \ dy = \lambda \int_{\Omega_p} u \ h \ dx \ dy \quad \text{for all } \ h \in V. \quad (8.2)
\]
This is the weak form of the eigenvalue problem of finding nontrivial solutions of the usual eigenvalue equation \( \Delta u = \lambda u \) on \( \Omega_p \) subject to the boundary conditions
\[
u = 0 \quad \text{on} \quad \partial \Omega_1 \times \Omega_2 \quad \text{and} \quad D_v u = 0 \quad \text{on} \quad \Omega_1 \times \partial \Omega_2 \quad (8.3)
\]

Suppose now that \( \mathcal{E}_1 := \{ e_{1j} : j \in \mathbb{N} \} \) is the family of Dirichlet Laplacian eigenfunctions on \( \Omega_1 \) defined as in Section 4 and \( \mathcal{F}_2 := \{ f_{2k} : k \in \mathbb{N} \} \) is the family of Neumann Laplacian eigenfunctions on \( \Omega_2 \) defined as in Section 6. Let \( \Lambda_1, \Lambda_2 \) be the associated sequences of eigenvalues and consider the family \( \mathcal{U} \) of dyads
\[
u_{jk} := e_{1j} \otimes f_{2k} \quad \text{defined on} \quad \Omega_1 \times \Omega_2; \ j, k \in \mathbb{N}. \quad (8.4)
\]
Each dyad \( \nu_{jk} \) is in \( V \) and the following holds.

**Theorem 8.1.** Assume \((B1), (B2)\) hold, and \( \Lambda_1, \Lambda_2, \mathcal{E}_1, \mathcal{F}_2, \mathcal{U} \) as above. Then each \( \nu_{jk} \in \mathcal{U} \) is a DNL eigenfunction of \((8.2)\) on \( \Omega_p \) corresponding to the eigenvalue \( \lambda_{1j} + \lambda_{2k} \). Moreover \( \mathcal{U} \) is an orthogonal family in \( V \) with respect to the inner product \([.,.]_{\nabla, \Omega_p}\) and is an orthogonal family in \( L^2(\Omega_p) \).

**Proof.** When \( h \in V \), then \( h(x, \cdot) \in H^1(\Omega_2) \) for a.e. \( x \in \Omega_1 \) and \( h(\cdot, y) \in H^1_0(\Omega_1) \) for a.e. \( y \in \Omega_2 \). From the Dirichlet eigenequation on \( \Omega_1 \),
\[
\int_{\Omega_1} f_{2k}(y) \nabla_x e_{1j}(x) \cdot \nabla_x h(x, y) \ dx = \lambda_{1j} f_{2k}(y) \int_{\Omega_1} e_{1j}(x) h(x, y) \ dx
\]
for almost all \( y \in \Omega_2 \). Integrating this over \( \Omega_2 \) then yields
\[
\int_{\Omega_2} \int_{\Omega_1} \nabla_x \nu_{jk}(x, y) \cdot \nabla_x h(x, y) \ dx \ dy = \lambda_{1j} \int_{\Omega_2} \int_{\Omega_1} \nu_{jk}(x, y) h(x, y) \ dx \ dy.
\]
Similarly one has that
\[
\int_{\Omega_1} \int_{\Omega_2} \nabla_y \nu_{jk}(x, y) \cdot \nabla_y h(x, y) \ dy \ dx = \lambda_{2k} \int_{\Omega_1} \int_{\Omega_2} \nu_{jk}(x, y) h(x, y) \ dy \ dx.
\]
Adding these shows that \( \nu_{jk} \) is a solution of \((8.2)\) with eigenvalue \( \lambda_{1j} + \lambda_{2k} \) as claimed.

The fact that the \( \nu_{jk} \) are \( \nabla \)-orthogonal follows by using Fubini’s theorem and the fact that the families \( \mathcal{E}_1, \mathcal{F}_2 \) are orthonormal with respect to the \( \nabla \)-inner product on \( H^1_0(\Omega_1), H^1(\Omega_2) \) respectively. These computations shows that the \( \nu_{jk} \) have
\[
[u_{jk}, u_{jk}]_{\nabla, \Omega_p} = \int_{\Omega_1} \int_{\Omega_2} \left[ |\nabla_x u_{jk}|^2 + |\nabla_y u_{jk}|^2 \right] dy \ dx = \frac{\lambda_{1j} + \lambda_{2k}}{1 + \lambda_{2k}}. \quad (8.5)
\]
They are $L^2$-orthogonal on $\Omega_p$ since the individual factors are $L^2$-orthogonal on $\Omega_1, \Omega_2$ respectively.

Note that the functions in $\mathcal{U}$ will be a maximal orthogonal set in $V$ from the general properties of tensor products of Hilbert spaces as $\mathcal{E}_1, \mathcal{F}_2$ are maximal orthogonal sets in $H^1_0(\Omega_1), H^1(\Omega_2)$ respectively. Therefore, the above result implies that the eigenfunctions of this DNL problem span $V$.

9. **Dirichlet-Robin Laplacian Eigenproblems on $\Omega_p$.**

A similar analysis holds when Dirichlet conditions are imposed on $\Omega_1$ and Robin conditions are required on $\Omega_2$. In this case take $V = H^1_0(\Omega_1) \otimes H^1(\Omega_2)$ with the inner product

$$[u, v]_{b_2} := \int_{\Omega_p} \nabla u \cdot \nabla v \, dx \, dy + \int_{\Omega_1} \int_{\partial \Omega_2} b_2(y) uv \, d\sigma_2 \, dx$$

This is easily verified to be an inner product when $b_2$ obeys (B3) on $\partial \Omega_2$.

The **Dirichlet Robin Laplacian (DRL) eigenproblem** on the product region $\Omega_p$ is the problem of finding nontrivial solutions $(\lambda, u) \in \mathbb{R} \times V$ of the system

$$[u, h]_{b_2} = \lambda \langle u, h \rangle_{2, \Omega_p} \quad \text{for all} \quad v \in V.$$  \hspace{1cm} (9.2)

This is the weak form of the eigenvalue problem of finding non-trivial solutions of the usual eigenvalue equation $\Delta u = \lambda u$ on $\Omega_p$ subject to the boundary conditions

$$u = 0 \quad \text{on} \quad \partial \Omega_1 \times \Omega_2 \quad \text{and} \quad D_\nu u + b_2 u = 0 \quad \text{on} \quad \Omega_1 \times \partial \Omega_2$$ \hspace{1cm} (9.3)

Suppose $\mathcal{E}_1 := \{e_{1j} : j \in \mathbb{N}\}$ is the family of Dirichlet Laplacian eigenfunctions on $\Omega_1$ defined as in Section 4 and that $\mathcal{G}_2 := \{g_{2k} : k \in \mathbb{N}\}$ is the family of Robin Laplacian eigenfunctions on $\Omega_2$ defined as in Section 7. Let $\Lambda_1, \Lambda_2$ be the associated sequences of eigenvalues and consider the family $\mathcal{U}$ of dyads

$$u_{jk} := e_{1j} \otimes g_{2k} \quad \text{with} \quad j, k \in \mathbb{N}.$$ \hspace{1cm} (9.4)

Each of these dyads is in $V$ and the following holds.

**Theorem 9.1.** Assume $\Omega_1, \Omega_2, b_2$ satisfy (B1)-(B3), and $\Lambda_1, \Lambda_2, \mathcal{E}_1, \mathcal{G}_2, \mathcal{U}$ as above. Then each $u_{jk} \in \mathcal{U}$ is a DRL eigenfunction of (9.2) on $\Omega_p$ corresponding to the eigenvalue $\lambda_{1j} + \lambda_{2k}$. Moreover $\mathcal{U}$ is an orthogonal family in $V$ with respect to the inner product $[\cdot, \cdot]_{b_2}$ and is an orthogonal family in $L^2(\Omega_p)$.

**Proof.** The fact that each $u_{jk} \in \mathcal{U}$ is a DRL eigenfunction of (9.2) on $\Omega_p$ with eigenvalue $\lambda_{1j} + \lambda_{2k}$ follows from the evaluation of the relevant integrals just as in the first part of the proof of Theorem 8.1. This leads to $[u_{jk}, h]_{b_2} = (\lambda_{1j} + \lambda_{2k}) \langle u_{jk}, h \rangle_{2, \Omega_p}$ for all $h \in V$. Take $h = u_{lm}$ and use Fubini’s theorem to verify that these functions are orthogonal in both the $L^2$- and $b_2$-inner products. Note that a formula similar to (8.5) can be obtained with the $b_2$-norm in place of $[u_{jk}, u_{jk}]_{\mathcal{F}_2}$. \hfill $\square$
10. Dirichlet-Steklov, Neumann-Steklov, and Robin-Steklov Laplacian Eigenproblems on \( \Omega \) and \( \Omega_p \).

The above sections all treated well-known eigenproblems for the Laplacian. Somewhat surprisingly, a similar result also holds for 2-parameter Dirichlet-Steklov (DSL), Neumann-Steklov (NSL) and Robin-Steklov (RSL) eigenproblems for the Laplacian. These are problems that involve an eigenparameter \( \lambda \) for the Laplacian and also a Steklov eigenvalue \( \delta \) in the boundary condition on \( \Omega_2 \). Such problems arise in fluid mechanics and elsewhere, see Auchmuty and Simpkins [6].

The weighted harmonic Steklov eigenproblem on a region \( \Omega \subset \mathbb{R}^N \) is the problem of finding non-trivial \( (\delta, s) \in \mathbb{R} \times H^1(\Omega) \) that satisfy the equation

\[
[s, h]_\nabla := \int_\Omega \nabla s \cdot \nabla h \, dx = \delta \int_{\partial \Omega} b \, s \, h \, d\sigma \quad \text{for all} \quad h \in H^1(\Omega). \tag{10.1}
\]

Here \( b \) is a function on \( \partial \Omega \) that obeys (B3). \( b \equiv 1/|\partial \Omega| \) is the standard harmonic Steklov eigenproblem.

This is the weak form of the problem of finding a solution of Laplace’s equation on \( \Omega \) subject to the boundary condition \( D_\nu s = \delta b \, s \) on \( \partial \Omega \). While this system was first studied by Steklov in 1902, the results used here are taken from Auchmuty [3] and [4] as well as Section 8 of [5]. Let \( \mathcal{H}(\Omega) \) be the orthogonal complement of \( H^1_0(\Omega) \), with respect to the inner product of (5.1). Then \( \mathcal{H}(\Omega) \) is the space of all \( H^1 \)-harmonic functions on \( \Omega \) when (B3) holds.

Observe that \( \delta_0 = 0 \) is an eigenvalue of (10.1) with the associated eigenfunction \( s_0(x) \equiv 1 \). This is the least eigenvalue and all other Steklov eigenvalues must be strictly positive. Consider the sequence of constrained variational problems \( P_k \) as described by (4.1) in [5] taking the bilinear forms \( a, m \) in that paper to be, respectively,

\[
[u, v]_{b,\Omega} := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} b \, u \, v \, d\sigma \quad \text{and} \quad \langle u, v \rangle_{b,\partial \Omega} := \int_{\partial \Omega} b \, u \, v \, d\sigma \tag{10.2}
\]

This generates a sequence of eigenfunctions \( s_k \) in \( H^1(\Omega) \) satisfying (10.1) that are normalized with respect to the inner product \( [\cdot, \cdot]_{b,\Omega} \). Let \( \mathcal{S} := \{ s_k : k \in \mathbb{N} \} \) be such a sequence and \( \Lambda := \{ \delta_k : k \in \mathbb{N} \} \) be the associated sequence of eigenvalues. Then the eigenequation (10.1) implies that

\[
[s_k, s_k]_\nabla = \frac{\delta_k}{1 + \delta_k} \quad \text{and} \quad \langle s_k, s_k \rangle_{b,\partial \Omega} = \frac{1}{1 + \delta_k} \quad \text{for all} \quad k \in \mathbb{N} \tag{10.3}
\]

\[
[s_j, s_k]_\nabla = 0 \quad \text{and} \quad \langle s_j, s_k \rangle_{b,\partial \Omega} = 0 \quad \text{when} \quad j \neq k.
\]

The orthonormal set \( \mathcal{S}_0 := \mathcal{S} \cup \{ s_0 \} \) in \( H^1(\Omega) \) is now a basis of the subspace \( \mathcal{H}(\Omega) \) of all \( H^1 \)-harmonic functions on \( \Omega \) as stated below.

**Theorem 10.1.** Assume that (B1)-(B3) hold and \( \Lambda, \mathcal{S} \) are sequences defined by the successive variational problems \( P_k \). Then \( \Lambda \) is an increasing sequence with \( \delta_k \to \infty \) and \( \mathcal{S}_0 \) is a maximal orthogonal set in \( \mathcal{H}(\Omega) \).
This is a combination of Theorem 8.2 of [5] and some analysis from [4]. The later paper shows that under further conditions on \( b \), the traces of the Steklov eigenfunctions on \( \partial \Omega \) are a maximal orthogonal set in \( L^2(\partial \Omega, b d\sigma) \).

The Dirichlet-Steklov Laplacian (DSL) eigenproblem on the product region \( \Omega_p := \Omega_1 \times \Omega_2 \) is the problem of finding nontrivial solutions \( (\lambda, \delta, u) \in \mathbb{R}^2 \times V \) of the system

\[
\int_{\Omega_p} \nabla u \cdot \nabla h \, dx \, dy = \lambda \int_{\Omega_p} u h \, dx \, dy + \delta \int_{\Omega} \int_{\partial \Omega_2} b_2 u h \, d\sigma_2 \, dx \quad \text{for all } h \in V. \tag{10.4}
\]

Here \( V := H^1_0(\Omega_1) \otimes H^1(\Omega_2) \) as in Section 8, \( b_2 \) is a function on \( \partial \Omega_2 \) that obeys (B3), and the left hand side of (10.4) is \([u, v]_{\nabla, \Omega_p}\).

This is a 2-parameter eigenproblem and is the weak form of the problem of finding nontrivial solutions of the system \(-\Delta u = \lambda u \) on \( \Omega_p \) subject to the boundary conditions

\[
u = 0 \text{ on } \partial \Omega_1 \times \Omega_2 \text{ and } D_n u = \delta b_2 u \text{ on } \Omega_1 \times \partial \Omega_2. \tag{10.5}
\]

Suppose now that \( \mathcal{E}_1 := \{e_{1j} : j \in \mathbb{N}\} \) and \( \Lambda_1 := \{\lambda_{1j} : j \in \mathbb{N}\} \) is the Dirichlet Laplacian eigendata on \( \Omega_1 \) defined as in Section 3, and let \( \mathcal{E}_2 := \{e_{2k} : k \in \mathbb{N}_0\} \) and \( \Lambda_2 := \{\delta_{2k} : k \in \mathbb{N}_0\} \) be the harmonic Steklov eigendata on \( \Omega_2 \) defined as above, with \( \Omega_2 \) in place of \( \Omega \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Let \( \mathcal{U} \) be the family consisting of the dyads

\[
u_{jk} := e_{1j} \otimes s_{2k} \quad \text{with } j \in \mathbb{N}, k \in \mathbb{N}_0. \tag{10.6}
\]

Then each of these dyads is in \( V \) and the following holds.

**Theorem 10.2.** Assume \( \Omega_1, \Omega_2 \) are regions that satisfy (B1) and (B2), \( b_2 \) satisfies (B3) on \( \partial \Omega_2 \), and define \( \Lambda_1, \Lambda_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{U} \) as above. Then each \( \nu_{jk} \in \mathcal{U} \) is a DSL eigenfunction of (10.4) on \( \Omega_p \) corresponding to the eigenpair \((\lambda_{1j}, \delta_{2k})\).

**Proof.** This result follows from a straightforward computation and the use of Fubini's theorem. When \( h \in V \), then \( h(x, \cdot) \in H^1(\Omega_2) \) for a.e. \( x \in \Omega_1 \) and \( h(\cdot, y) \in H^1_0(\Omega_1) \) for a.e. \( y \in \Omega_2 \). From the Dirichlet eigenequation on \( \Omega_1 \),

\[
\int_{\Omega_1} s_{2k}(y) \nabla_x e_{1j}(x) \cdot \nabla_x h(x, y) \, dx = \lambda_{1j} s_{2k}(y) \int_{\Omega_1} e_{1j}(x) h(x, y) \, dx
\]

for almost all \( y \in \Omega_2 \). Integrating this over \( \Omega_2 \), then yields

\[
\int_{\Omega_p} \nabla_x \nu_{jk}(x, y) \cdot \nabla_x h(x, y) \, dx \, dy = \lambda_{1j} \int_{\Omega_p} \nu_{jk}(x, y) h(x, y) \, dx \, dy.
\]

Similarly, from the Steklov eigenequation on \( \Omega_2 \) one has that

\[
\int_{\Omega_p} \nabla_y \nu_{jk}(x, y) \cdot \nabla_y h(x, y) \, dy \, dx = \delta_{2k} \int_{\Omega_1} \int_{\partial \Omega_2} b_2(y) \nu_{jk}(x, y) h(x, y) \, d\sigma_2 \, dx.
\]

Adding these shows that \( \nu_{jk} \) is a solution of (10.4) with eigenpair \((\lambda_{1j}, \delta_{2k})\) as claimed. \( \square \)

From the preceding Theorems 4.1 and 10.1, the sets \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are maximal orthonormal sets in \( H^1_0(\Omega) \) and \( \mathcal{H}(\Omega) \) so they will be a maximal linearly independent set in the space \( H^1_0(\Omega) \otimes \mathcal{H}(\Omega) \). Unfortunately the computation of \([\nu_{jk}, u_{lm}]_{\nabla, \Omega_p}\) with the formulae from the above proof, shows that this inner product on \( V \) may be nonzero when \( j = l \) and \( k \neq m \) since one has that \( s_{2k}, s_{2m} \) are not, in general, \( L^2 \)-orthogonal on \( \Omega_2 \).
The Neumann-Steklov Laplacian (NSL) eigenproblem on the product region \( \Omega_p := \Omega_1 \times \Omega_2 \) is the problem of finding nontrivial solutions \((\lambda, \delta, u) \in \mathbb{R}^2 \times H^1(\Omega_p)\) of the system

\[
\int_{\Omega_p} \nabla u \cdot \nabla h \, dx \, dy = \lambda \int_{\Omega_p} u \, h \, dx \, dy + \delta \int_{\Omega_1} \int_{\partial\Omega_2} b_2 u \, h \, d\sigma_2 \, dx. \tag{10.7}
\]

for all \( h \in H^1(\Omega) \). This is the weak form of the eigenvalue problem of finding nontrivial solutions of the system \(-\Delta u = \lambda u\) on \(\Omega_p\) subject to the boundary conditions

\[
D_v u = 0 \quad \text{on} \quad \partial\Omega_1 \times \Omega_2 \quad \text{and} \quad D_v u = \delta \, b_2 u \quad \text{on} \quad \Omega_1 \times \partial\Omega_2. \tag{10.8}
\]

Suppose now that \( F := \{f_{ij} : j \in \mathbb{N}\} \) and \( \Lambda := \{\lambda_{1j} : j \in \mathbb{N}\} \) is the Neumann Laplacian eigendata on \(\Omega_1\) defined as in Section 6, and \( S_2, \Lambda_2 \) is the harmonic Steklov eigendata on \(\Omega_2\) defined as above. Let \( U \) be the family consisting of the dyads

\[
u_{jk} := f_{ij} \otimes s_{2k} \quad \text{with} \quad j \in \mathbb{N}, \ k \in \mathbb{N}_0,
\tag{10.9}
\]

Each of these dyads is in \( H^1(\Omega_p) \) and the following holds.

**Theorem 10.3.** Assume \( \Omega_1, \Omega_2 \) are regions that satisfy \((B1)\) and \((B2)\), \( b_2 \) satisfies \((B3)\) on \( \partial\Omega_2 \), and define \( \Lambda_1, \Lambda_2, F_1, S_2, U \) as above. Then each \( \nu_{jk} \in U \) is an NSL eigenfunction of \((10.7)\) on \(\Omega_p\) corresponding to the eigenpair \((\lambda_{1j}, \delta_k)\).

**Proof.** This result follows from a straightforward computation and the use of Fubini’s theorem. When \( h \in H^1(\Omega_p) \), then \( h(x, \cdot) \in H^1(\Omega_2) \) for a.e. \( x \in \Omega_1 \) and \( h(\cdot, y) \in H^1(\Omega_1) \) for a.e. \( y \in \Omega_2 \). From the Neumann eigenequation on \(\Omega_1\),

\[
\int_{\Omega_1} s_{2k}(y) \nabla_x f_{ij}(x) \cdot \nabla_x h(x, y) \, dx = \lambda_{1j} s_{2k}(y) \int_{\Omega_1} f_{ij}(x) \, h(x, y) \, dx
\]

for almost all \( y \in \Omega_2 \). Integrating this over \(\Omega_2\), then yields

\[
\int_{\Omega_p} \nabla_x u_{jk}(x, y) \cdot \nabla_x h(x, y) \, dx \, dy = \lambda_{1j} \int_{\Omega_p} u_{jk}(x, y) \, v(x, y) \, dx \, dy.
\]

Similarly one has that

\[
\int_{\Omega_p} \nabla_y u_{jk}(x, y) \cdot \nabla_y h(x, y) \, dy \, dx = \delta_{2k} \int_{\Omega_1} \int_{\partial\Omega_2} b_2(y) \, u_{jk}(x, y) \, h(x, y) \, d\sigma_2 \, dx.
\]

Adding these shows that \( u_{jk} \) is a solution of \((10.4)\) with eigenpair \((\lambda_{1j}, \delta_{2k})\) as claimed. \( \square \)

From the preceding Theorems 6.1 and 10.1, the sets \( F_1 \) and \( S_2 \) are maximal orthonormal sets in \( H^1(\Omega_1) \) and \( H(\Omega_2) \) so they will be a maximal linearly independent set in the space \( H^1(\Omega) \otimes H(\Omega) \). Again inner products of the form \([u_{jk}, u_{lm}]_{\nabla, \Omega_p}\) need not always be zero when \( k \neq m \) so there are only partial orthogonality results in this case.

The Robin-Steklov Laplacian (RSL) eigenproblem on the product region \( \Omega_p := \Omega_1 \times \Omega_2 \) is the problem of finding nontrivial solutions \((\lambda, \delta, u) \in \mathbb{R}^2 \times H^1(\Omega_p)\) of the system

\[
\int_{\Omega_p} \nabla u \cdot \nabla h \, dx \, dy + \int_{\Omega_2} \int_{\partial\Omega_1} b_1 u_{jk} h \, d\sigma_1 \, dy = \lambda \int_{\Omega_p} u \, h \, dx \, dy + \delta \int_{\Omega_1} \int_{\partial\Omega_2} b_2 u \, h \, d\sigma_2 \, dx \tag{10.10}
\]
This is the weak form of the eigenvalue problem of finding nontrivial solutions of the system $-\Delta u = \lambda u$ on $\Omega_p$ subject to the boundary conditions

$$D_\nu u + b_1 u = 0 \quad \text{on} \quad \partial \Omega_1 \times \Omega_2 \quad \text{and} \quad D_\nu u = \delta b_2 u \quad \text{on} \quad \Omega_1 \times \partial \Omega_2. \quad (10.11)$$

Suppose now that $G_1 := \{g_{1j} : j \in \mathbb{N}\}$ and $\Lambda_1 := \{\lambda_{1j} : j \in \mathbb{N}\}$ is the Robin Laplacian eigendata on $\Omega_1$ defined as in Section 7, and $S_2, \Lambda_2$ is the harmonic Steklov eigendata defined as above. Let $U$ be the family consisting of the dyads

$$u_{jk} := g_{1j} \otimes s_{2k} \quad \text{with} \quad j \in \mathbb{N}, k \in \mathbb{N}_0. \quad (10.12)$$

Each of these dyads is in $H^1(\Omega_p)$ and the following holds.

**Theorem 10.4.** Assume $\Omega_1, \Omega_2$ are regions that satisfy (B1) and (B2), $b$ satisfies (B4) on $\partial \Omega_p$, and $\Lambda_1, \Lambda_2, G_1, S_2, U$ as above. Then each $u_{jk} \in U$ is a RSL eigenfunction of (10.10) corresponding to the eigenpair $(\lambda_{1j}, \delta_{2k})$.

**Proof.** As in the previous theorems, this result follows from a straightforward computation and the use of Fubini’s theorem. When $h \in H^1(\Omega_p)$, then $h(x, \cdot) \in H^1(\Omega_2)$ for a.e. $x \in \Omega_1$ and $h(\cdot, y) \in H^1(\Omega_1)$ for a.e. $y \in \Omega_2$. The Robin eigenequation on $\Omega_1$ holds for $u = g_{1j}$, $\lambda = \lambda_{1j}$ and test function $h(\cdot, y)$ for almost every $y \in \Omega_2$. Multiply that equation by $s_{2k}$ and integrate over $\Omega_2$ to obtain

$$\int_{\Omega_p} \nabla_x u_{jk} \cdot \nabla_x h \, dx \, dy + \int_{\Omega_2} \int_{\partial \Omega_1} b_1 u_{jk} h \, d\sigma_1 \, dy = \lambda_{1j} \int_{\Omega_p} u_{jk} h \, dx \, dy. \quad (10.13)$$

Similarly from the Steklov eigenequation on $\Omega_2$ one obtains

$$\int_{\Omega_p} \nabla_y u_{jk} \cdot \nabla_y h \, dx \, dy = \delta_{2k} \int_{\Omega_1} \int_{\partial \Omega_2} b_2 u_{jk} h \, d\sigma_2 \, dx. \quad (10.14)$$

Adding these shows that $u_{jk}$ is a solution of (10.10) with eigenpair $(\lambda_{1j}, \delta_{2k})$ as claimed. $\square$

From the preceding Theorems 7.1 and 10.1, the sets $G_1$, and $S_2$ are maximal orthonormal sets in $H^1(\Omega_1)$ and $\mathcal{H}(\Omega_2)$ so they will be a maximal linearly independent set in the space $H^1(\Omega_1) \otimes \mathcal{H}(\Omega_2)$.

**11. Boundary Trace Spaces for $\Omega_p$.**

A well-known characterization of the trace space $H^{1/2}(\partial \Omega)$ of a region $\Omega \subset \mathbb{R}^N$ is that it is isomorphic to the quotient space $H^1(\Omega)/H^1_0(\Omega)$ or, equivalently, to the orthogonal complement of $H^1_0(\Omega)$ in $H^1(\Omega)$. Here we shall describe this orthogonal complement when $\Omega$ is a product region and, in particular, obtain orthogonal bases of this space.

The definition of the space $\mathcal{H}(\Omega)$ implies that

$$H^1(\Omega) = H^1_0(\Omega) \oplus_b \mathcal{H}(\Omega) \quad (11.1)$$

when $\Omega$ satisfies (B1) and (B3) holds. Thus the harmonic Steklov eigenfunctions on a region provide a basis for the trace space of $H^1-$functions. See Auchmuty [4] for an analysis of this situation. In view of this one can ask about possible decompositions of the space of harmonic functions on $\Omega_p$ in terms of functions on $\Omega_1, \Omega_2$. 
A simple calculation shows that the tensor product of two Steklov eigenfunctions on regions need not be a Steklov eigenfunction on the product region. Also there are Steklov eigenfunctions on product regions that are not obtained as finite linear combinations of Steklov eigenfunctions on the individual components. When $\Omega \subset \mathbb{R}$, then there are only two Steklov eigenfunctions of the Laplacian on $\Omega$ while there are infinitely many Steklov eigenfunctions on a box. See Auchmuty and Cho [7], or Girouard and Polterovich [10] for explicit formulae for the harmonic Steklov eigenfunctions on boxes in the plane.

The preceding results lead to the following isomorphism for the space of harmonic functions on $\Omega_p$ when $b \equiv 1$ on $\partial \Omega$.

**Theorem 11.1.** Suppose that $\Omega_p := \Omega_1 \times \Omega_2$ and both $\Omega_1, \Omega_2$ satisfy $(B2)$, $b \equiv 1$ on $\partial \Omega_p$ and $H^1(\Omega_p)$ has this $b$-inner product. Then $\mathcal{H}(\Omega_p)$ is linearly isomorphic to $V_0 \oplus V_1 \oplus V_2$ where

$$V_0 = \mathcal{H}(\Omega_1) \otimes \mathcal{H}(\Omega_2), \quad V_1 = H_0^1(\Omega_1) \otimes \mathcal{H}(\Omega_2), \quad V_2 = \mathcal{H}(\Omega_1) \otimes H_0^1(\Omega_2). \quad (11.2)$$

Moreover if $S_1, S_2$ are orthonormal bases of $\mathcal{H}(\Omega_1), \mathcal{H}(\Omega_2)$ respectively, and $E_1, E_2$ are orthonormal bases of $H_0^1(\Omega_1), H_0^1(\Omega_2)$ as in Section 4, then the sets $S_1 \otimes S_2, E_1 \otimes S_2, S_1 \otimes E_2$, are maximal orthogonal sets in $V_0, V_1, V_2$ respectively.

**Proof.** From Theorem 3.2 upon using (11.1) for $\Omega_1$ and $\Omega_2$ as well as (11.1) for the region $\Omega_p$, one sees that

$$H^1(\Omega_p) = H_0^1(\Omega_p) \oplus V \quad \text{with} \quad V = V_0 \oplus V_1 \oplus V_2,$$

with this $b$-inner product being used. Thus $V$ and $\mathcal{H}(\Omega_p)$ are linearly isomorphic as they are orthogonal complements of $H_0^1(\Omega_p)$ in $H^1(\Omega_p)$. The characterization of the bases of $V_0, V_1, V_2$ follows from the properties of bases of Hilbert tensor products. $\Box$

This construction of an orthonormal basis for $\mathcal{H}(\Omega_p)$ may be used to provide a different description of the boundary traces of $H^1$-functions on $\Omega_p$. It implies that any function with nonzero boundary trace can be uniquely written as a function in $V$ and one with zero trace. Thus the $H^1$-boundary trace space is linearly isomorphic to $V$.

**References**


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