1. Is the set $V$ of positive real numbers (i.e. $x > 0$) with addition and scalar multiplication defined as follows:

$$u + v = uv, \quad cu = uc,$$

a vector space? Why?

*Answer:* It is a vector space due to all 8 axioms!

2. Is the set $V$ of two-dimensional vectors $v$ with the standard scalar multiplication and vector addition defined by

$$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + 2v_1, u_2 + v_2)$$

a vector space? Why?

*Answer:* It is not a space since the commutative law of addition does not hold.

3. Consider the set $W = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0\}$. Is $W$ a subspace of $\mathbb{R}^3$?

*Answer:* Yes, it is a subspace due to 3 properties of it.

4. Let $M$ be the space of $2 \times 2$ matrices. Consider the set $W = \left\{A \in M : A = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix}\right\}$. Is $W$ a subspace of $M$?

*Answer:* No, it is not a subspace since the zero element is not in it.

5. Find all special solutions to

$$\begin{cases}
x_1 - x_2 + 2x_3 = 0 \\
2x_1 + x_2 + x_3 = 0 \\
x_1 + x_2 = 0
\end{cases}$$

*Answer:* $[R|0] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ hence $s = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

6. Construct a nullspace matrix $N$ for the matrix $A$ such that its reduced row echelon matrix is:

(a) $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

*Answer:* since the number of special solutions is $n - r = 2 - 2 = 0$ hence $N$ is empty.

(b) $R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

*Answer:* $N = \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ since the number of columns in $N$ is $n - r = 4 - 2 = 2$.
7. For matrices

\[
A = \begin{bmatrix}
2 & 6 & 4 \\
-1 & 2 & 3 \\
3 & -1 & -4 \\
4 & 3 & 5 \\
2 & 4 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 9 & 1 & -3 & 6 \\
-1 & -6 & 1 & 1 & -2 \\
3 & -2 & 1 & 4 & 1 \\
1 & 3 & 0 & 2 & 4 \\
0 & 2 & 0 & 1 & 3 \\
3 & 1 & 1 & 2 & 2
\end{bmatrix}
\]

describe their column and null-spaces.

Answer: \(R_A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}\) hence \(N(A) = \{0\} \subset \mathbb{R}^3\)

\[
R_B = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}\) hence \(N(B) = \{x \in \mathbb{R}^5 : x = x_6 \begin{bmatrix}
1 \\
-1 \\
-2 \\
-1 \\
1
\end{bmatrix}\} \}

8. True or False (with reason if true and example if false):

(a) \(C(A) = \mathbb{R}^2\) for \(A = \begin{bmatrix}
1 & -1 \\
1 & 1 \\
1 & 2
\end{bmatrix}\) ?

Answer: F

(b) \(N(A) = \{0\}\) for \(A = \begin{bmatrix}
1 & 2 & -1 \\
1 & 1 & 1
\end{bmatrix}\) ?

Answer: F

(c) Vectors \(\begin{bmatrix}1 \\ 1 \end{bmatrix}, \begin{bmatrix}2 \\ 1 \end{bmatrix}, \begin{bmatrix}0 \\ 1 \end{bmatrix}, \begin{bmatrix}-1 \\ 1 \end{bmatrix}\) are linearly independent?

Answer: F

(d) The reduced row echelon matrix for \(A = \begin{bmatrix}
1 & -1 \\
1 & 1 \\
1 & 2
\end{bmatrix}\) is \(R = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}\) ?

Answer: T

(e) The rank of \(A\) from (b) is \(r = 2\)?

Answer: T

(f) If \(n > m\) then there are at least one “free” variable?

Answer: T

(g) The number of special solutions to \(Ax = 0\) with \(A \in \mathbb{R}^{m \times n}\) is in general equal to \(m\)?

Answer: F

(h) \(C(A) = C(R)\) where \(R\) is the reduced row echelon matrix for \(A\)?

Answer: F

(i) \(N(A) = C(R)\) where \(R\) is the reduced row echelon matrix for \(A\)?

Answer: F

(j) The rank of \(A\) is the number of nonzero rows in \(R\)?

Answer: T

(k) \(C(A) = C(2A)\)?
9. Is the following set of vectors linearly dependent or independent:

\[ v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -6 \\ 7 \\ 3 \\ -1 \\ 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \text{hence linearly independent} \]

**Answer:** T