Homework #1

due Thursday, February 10, 2011

Norms in $\mathbb{R}^N$, Review of Continuous Functions, Minimizers.

1. Prove that if a sequence in $\mathbb{R}^N$ is bounded then it has a convergent subsequence.

2. (i) Prove that if $f$ is a continuous mapping of a metric space $(X, \rho_X)$ into a metric space $(Y, \rho_Y)$ then $f(E) \subseteq \overline{f(E)}$ for every set $E \subseteq X$ (where $E$ is the closure of $E$).
   (ii) Prove that if $f$ is continuous and $E$ is compact then $f(E) = \overline{f(E)}$.
   (iii) Construct an example showing that $f(E) \subseteq \overline{f(E)}$ may not be true if $f$ is not continuous.

3. Construct an example of a function in $\mathbb{R}$ for each of the following situations:
   (i) A continuous function $f$ and a bounded set $E$ with $f(E)$ not bounded.
   (ii) A continuous function $f$ and an open set $E$ with $f(E)$ not open.
   (iii) A continuous function $f$ and a closed set $E$ with $f(E)$ not closed.

4. Show that $\rho(x, y) := |\ln(y/x)|$ is a metric on $\mathbb{R}^+$. Is $(\mathbb{R}^+, \rho)$ a complete metric space?

5. Prove the following theorem:
   For each $x \in \mathbb{R}^n \setminus \{0\}$, $p > 0$, $\|x\|_p$ is a decreasing function of $p$. It’s strictly decreasing when $x$ has more than 1 non-zero component. Moreover,
   $$\lim_{p \to \infty} \|x\|_p = \inf_{p \geq 1} \|x\|_p = \|x\|_\infty.$$

6. For $1 \leq p < q \leq \infty$ and $x \in \mathbb{R}^n \setminus \{0\}$ prove that
   $$0 \leq \|x\|_q \leq \|x\|_p \leq n^{1/p-1/q} \|x\|_q.$$

7. Let $(X, \rho)$ be a metric space and $f : X \to \mathbb{R}$ be a function. Then $f$ is l.s.c. on $X$ if and only if the synoptic sets $S_c(f) = \{x \in X : f(x) \leq c\}$ are closed for each $c \in \mathbb{R}$.

8. For $m \in \mathbb{N}$ define $f_m(x) = \frac{x}{m} e^{-x/m}$ for $x \geq 0$ and $G(x) = \sup_{m \in \mathbb{N}} f_m(x)$. Show that $G(x)$ is continuous and bounded on $[0, \infty)$. What is $\lim_{x \to \infty} G(x)$?

9. Let $g(t)$ be a function on $0 \leq t \leq \infty$ such that $\lim_{t \to \infty} g(t) = +\infty$. Let $f$ be a continuous function on a closed non-empty set $E$ such that $f(x) \geq g(|x|)$. Show that there is a point $x_0 \in E$ that minimizes $f$ on $E$.

10. Suppose that $f(x) = x^2 + 2a \cos x - 2bx$ with $a, b$ constants. Show that this function has minimizers on $\mathbb{R}$ and find the equations that they satisfy. Find bounds on this minimizer and show that the minimum value is less than or equal to $(a + 1)^2 - b^2 - 1$.

11. Consider $f$ defined in #10. Show that there is a unique critical point of this function when $|a| < 1$. Show that there are multiple critical points of this function when $|a| > 1$. 

1