Integral Solution, Rankine-Hugoniot Condition
Introduction to 1-dim Conservation Laws

Lecture 21

November 07, 2013
Consider the IVP for the 1-dim scalar conservation laws:

\[
\begin{align*}
    u_t + [F(u)]_x &= 0, \quad t \geq 0, \quad x \in \mathbb{R} \\
    u(0, x) &= g(x) \quad x \in \mathbb{R}
\end{align*}
\]  

(1)

where \( F(\cdot) \) is a given function, and \( g \) is a prescribed initial data.

Problem (1) is also called the **Cauchy problem**

**Example - Burgers’ Equation**

When \( F(u) = u^2/2 \), problem (1) is called the **Burgers’ equation**:

\[
\begin{align*}
    u_t + uu_x &= 0, \quad t \geq 0, \quad x \in \mathbb{R} \\
    u(0, x) &= g(x) \quad x \in \mathbb{R}
\end{align*}
\]  

(2)
Integral Solution of (1)

Suppose \( u(t, x) \) is a smooth solution of (1), multiply this equation by a smooth "test function" \( v(t, x) \) with compact support, i.e. \( v \in C^1_c([0, \infty) \times \mathbb{R}) \), and integrate by parts:

\[
0 = \int_0^\infty \int_{-\infty}^\infty v(t, x) (u_t + [F(u)]_x) \, dx \, dt \\
= -\int_0^\infty \int_{-\infty}^\infty [v_t u + F(u)v_x] \, dx \, dt + \int_{-\infty}^\infty \left( \int_0^\infty v(t, x)u(0, x) \, dt \right) \, dx \\
= -\int_0^\infty \int_{-\infty}^\infty [v_t u + F(u)v_x] \, dx \, dt + \int_{-\infty}^\infty \left( \int_0^\infty v(t, x)g(x) \, dt \right) \, dx \tag{3}
\]

(the terms at \( x = \pm \infty \) vanish since \( v(t, x) \) has compact support)

Definition

When (3) holds true for all \( v \in C^1_c([0, \infty) \times \mathbb{R}) \), the function \( u(t, x) \) is called an integral solution of (1).
Consider a simple p.w. constant solution \( u(t, x) \) of (1) such that

\[
   u(t, x) = \begin{cases} 
   u_\ell, & x < \gamma(t) \\
   u_r, & x > \gamma(t) 
\end{cases}
\]  

(4)

where \( u_\ell \) and \( u_r \) are constants.

**Question:** How should the point \( \gamma(t) \) evolve for \( u(t, x) \) to be an integral solution of (1)?

Take \( v \in C^1([0, \infty) \times \mathbb{R}) \) that vanishes at \( t = 0 \), then (3) yields

\[
0 = \int_0^\infty \int_{-\infty}^{\infty} [v_t u + F(u) v_x] \, dx \, dt \\
= \int_0^\infty \int_{-\infty}^{\gamma(t)} [v_t u_\ell + F(u_\ell) v_x] \, dx \, dt + \int_0^\infty \int_{\gamma(t)}^\infty [v_t u_r + F(u_r) v_x] \, dx \, dt
\]  

(5)
First, look at
\[ \int_0^\infty \int_{-\infty}^{\gamma(t)} F(u_\ell) v_x \, dx \, dt + \int_0^\infty \int_{\gamma(t)}^\infty F(u_r) v_x \, dx \, dt \]
\[ = \int_0^\infty [F(u_\ell) - F(u_r)] v(t, \gamma(t)) \, dt, \]  
(6)

and
\[ \int_0^\infty \int_{-\infty}^{\gamma(t)} u_\ell v_t \, dx \, dt \]
\[ = \int_0^\infty \frac{d}{dt} \left[ \int_{-\infty}^{\gamma(t)} u_\ell v(t, x) \, dx \right] \, dt - \int_0^\infty u_\ell v(t, \gamma(t)) \dot{\gamma}(t) \, dt \]

Since \( v(0, x) = 0 \) for all \( x \in \mathbb{R} \), and \( v(t, x) = 0 \) for all \( t \geq T \) for some \( T > 0 \), the first term of the RHS vanishes, hence,
\[ \int_0^\infty \int_{-\infty}^{\gamma(t)} u_\ell v_t \, dx \, dt = -\int_0^\infty u_\ell v(t, \gamma(t)) \dot{\gamma}(t) \, dt \]  
(7)
Similarly, one has

\[ \int_0^\infty \int_{\gamma(t)}^\infty u_r v_t \, dx \, dt = \int_0^\infty u_r v(t, \gamma(t)) \dot{\gamma}(t) \, dt \]  \tag{8} 

Plugging (6), (7), (8) into (5), obtain

\[ 0 = \int_0^\infty [u_r - u_\ell] v(t, \gamma(t)) \dot{\gamma}(t) \, dt + \int_0^\infty [F(u_\ell) - F(u_r)] v(t, \gamma(t)) \, dt \]

that holds for all \( v \in C^1([0, \infty) \times \mathbb{R}) \), hence, derive

**Rankine-Hugoniot (RH) Condition**

\[ \dot{\gamma}(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r} \]  \tag{9}
The RH condition can be generalized to solutions that are not p.w. constant but p.w. smooth. For this consider an integral solution $u(t, x)$ of (1), s.t. $u(t, x)$ is smooth at $x < \gamma(t)$, and at $x > \gamma(t)$ but has a jump discontinuity at $x = \gamma(t)$:

$$
\begin{cases}
    u_\ell(t, x), & x < \gamma(t) \\
    u_r(t, x), & x > \gamma(t)
\end{cases}
$$

(10)

Denote: $u^+(t) = \lim_{x \to \gamma(t)^+} u(t, x)$, $u^-(t) = \lim_{x \to \gamma(t)^-} u(t, x)$

**Theorem (Rankine-Hugoniot Condition)**

Let $u(t, x)$ given by (10) is an integral solution of (1). Then

$$
\dot{\gamma}(t) = \frac{F(u^-(t)) - F(u^+(t))}{u^-(t) - u^+(t)}, \quad t \geq 0
$$
References

- Evans pp. 135–139