Lax-Oleinik Formula
for Solution of Burgers Equation
Introduction to 1-dim Conservation Laws

Lecture 23

November 14, 2013
Consider the Burger’s equation

\[ u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R} \tag{1} \]

with initial condition:

\[ u(0, y) = g(y), \quad y \in \Gamma, \tag{2} \]

on the surface of initial data \( \Gamma \), where we assume that

\[ \exists M > 0 : \quad |g(x)| \leq M, \quad \forall x \in \mathbb{R} \]

Introduce a function

\[ s(t, x; y) = \frac{t}{2} \left( \frac{x - y}{t} \right)^2 + h(y) \tag{3} \]

with

\[ h(y) = \int_0^y g(z) \, dz \]
Then the function \( s(t, x; y) \) is minimized over \( y \) when

\[
x = y + tg(y)
\]

Moreover, \( s(t, x; y) \) is weakly coercive and continuous in \( y \) for all \( t \) and \( x \) fixed, hence, it attains its \text{global minimum} at some points \( y \). Denote by \( y(t, x) \) the \text{smallest} such point.

Note that \( y(t, x) \) is non-decreasing in \( x \), hence it has at most countably many discontinuities and is differentiable in \( x \) a.e.

Set now

\[
w(t, x) = \frac{t}{2} \left( \frac{x - y(t, x)}{t} \right)^2 + h(y(t, x)) = \inf_{y \in \mathbb{R}} s(t, x; y) \quad (4)
\]
Then \( w(t, x) \) is differentiable in \( x \) a.e. Moreover, it is also Lipschitz continuous in \( x \).

Also, note that as \( t \to 0 \) one has \( y(t, x) \to x \). Therefore, we have

\[
w(t, x) \to h(x), \quad t \to 0
\]

The function \( y(t, x) \) where it is differentiable, satisfies

\[
y_x + tg'(y)y_x = 1,
y_t + g(y) + tg'(y)y_t = 0
\]

that yield

\[
y_t = -g(y)y_x
\]
Now compute

\[ w_t = -\frac{1}{2t^2}(x - y(t, x))^2 + \frac{y(t, x) - x}{t} y_t + h'(y(t, x)) y_t, \]
\[ w_x = \frac{1}{t}(x - y(t, x))(1 - y_x) + h'(y(t, x)) y_x \]

then

\[ w_t = -\frac{g^2(y(t, x))}{2}, \]
\[ w_x = g(y(t, x)) \]

Hence, \( w(t, x) \) is Lipschitz continuous a.e., and wherever it is differentiable, it solves the IVP

\[ w_t + \frac{w_x^2}{2} = 0, \quad w(0, x) = h(x) \quad (5) \]
Define a.e.

Lax-Oleinik Formula for Burger’s equation

\[ u(t, x) = \frac{\partial w}{\partial x}(t, x) = \frac{\partial}{\partial x} \left[ \frac{t}{2} \left( \frac{x - y(t, x)}{t} \right)^2 + h(y(t, x)) \right] \] (6)

Function (6) is an integral solution for (1)-(2)!
References

- Evans pp. 143–148