Completeness, Compactness

Week 03

September 8–12, 2014
Completeness

Definition (Cauchy sequence)

A sequence \( \{x_n\} \subset X \) is a Cauchy sequence iff \( \rho(x_n, x_m) \to 0 \) as \( n, m \to \infty \).

Obviously, if \( \{x_n\} \) is convergent then it is Cauchy.

Lemma

If \( \{x_n\} \) is a Cauchy sequence in \( (X, \rho) \) then it is bounded.

Examples:

1. \( (\mathbb{Q}, |r_1 - r_2|) \), \( \{r_n\} = \left\{ \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n} \right\} \Rightarrow r_n \) is both Cauchy and convergent (to \( r_0 = 0 \)). Now take \( \{r_n\} = \left(1 + \frac{1}{n}\right)^n \) which is Cauchy but not convergent in \( \mathbb{Q} \) since \( \lim_{n \to \infty} r_n = e \notin \mathbb{Q} \).

2. \( \left(\mathbb{P}(t), \max_{t \in [0,1]} |P(t) - Q(t)| \right) \), where \( \mathbb{P}(t) \) is the space of polynomials \( P(t), 0 \leq t \leq 1 \). Let \( \{P_n(t)\} \) be a sequence that uniformly convergent to a continuous function which in its turn not a polynomial. Obviously, \( \{P_n(t)\} \) is Cauchy, but does not have a limit in \( \mathbb{P}(t) \).
Completeness (cont.)

Definition (complete metric space)

If every Cauchy sequence in a metric space $X$ is convergent then $X$ is called complete.

Examples:

1. Euclidean space $\mathbb{R}^n$ is complete.
2. $(\mathbb{Q}, \rho_2)$ is not complete.
3. Space $C[0,1]$ is complete.
4. Space $m$ of all bounded functions is complete.
5. Spaces $L_p[0,1]$ and $\ell_p$ are complete.
6. $\left( \mathbb{P}(t), \max_{t \in [0,1]} |P(t) - Q(t)| \right)$ is not complete.
7. $\left( C[0,1], \rho_2 := \left[ \int_0^1 |x(t) - y(t)|^2 dt \right]^{1/2} \right)$ is not complete.
Completeness (cont.)

**Definition (complete subspace)**

A subset $Y$ of a metric space $(X, \rho)$ is **complete** if the metric space $(Y, \rho|_Y)$ is complete.

**Theorem**

*Let $(X, \rho)$ be a complete metric space, and $(Y, \rho)$ be a subspace of $(X, \rho)$. Then $(Y, \rho)$ is complete if $Y$ is a closed set in $(X, \rho)$.*

\[ \square \ (\Rightarrow) \ (Y, \rho) \text{ is complete. To show: } Y \text{ contains all its accumulation points?} \]

Let $y \in Y$ is accumulation pt $\Rightarrow \exists \{y_n\} \in Y \setminus \{y\}$ s.t. $y_n \in B_{\frac{1}{n}}(y) \Rightarrow$

$\rho(y_n, y) < \frac{1}{n} \rightarrow 0 \Rightarrow \{y_n\}$ is convergent in $(X, \rho) \Rightarrow \{y_n\}$ is Cauchy $\Rightarrow$ (b/c $Y$ is complete) $\exists y_0 \in Y : y_n \rightarrow y_0 \in Y$. By uniqueness of lim: $y_0 = y \Rightarrow y \in Y$

\[ \square \ (\Leftarrow) \ Y \text{ is closed in } (X, \rho). \text{ Consider } \{y_n\} \text{ is Cauchy in } (Y, \rho) \Rightarrow \{y_n\} \text{ is Cauchy in } (X, \rho) \Rightarrow y_n \rightarrow y_0 \in X \Rightarrow \text{ (from Closed Set Thm) } y_0 \in Y \Rightarrow y_n \text{ converges to } y_0 \in Y \text{ in } (Y, \rho) \Rightarrow (Y, \rho) \text{ is complete} \square \]
Isometry & Isomorphism

Definition (isometry)

A map $\phi : X \to Y$ is called an isometry or an isometric embedding of $X$ into $Y$ iff

$$\rho_Y(\phi(x), \phi(y)) = \rho_X(x, y) \quad \forall x, y \in X$$

i.e. isometry is 1-to-1 and continuous

Examples:

Reflections, rotations, translations are isometries

Definition (isomorphism)

An isometry which is also onto is called an isomorphism. Two metric spaces $X$ and $Y$ are isomorphic iff there is an isomorphism $\phi : X \to Y$.

Examples:

1. the space of complex numbers $\mathbb{C}$ and $\mathbb{R}^2$
2. the space of quadratic polynomials $\mathbb{P}_2$ and $\mathbb{R}^3$
Completion of a metric space

Definition (completion of a metric space)

A metric space \((\overline{X}, \overline{\rho})\) is called the completion of \((X, \rho)\) iff

1. \(\exists\) an isometric embedding \(\phi : X \rightarrow \overline{X}\);
2. the image space \(\phi(X)\) is dense in \(\overline{X}\);
3. the space \((\overline{X}, \overline{\rho})\) is complete

Theorem

Every metric space has a completion, which is unique up to isomorphism

Proof in NH pp. 19–22
Compactness

**Definition (precompact set)**

A subset $K$ of a metric space $X$ is called **precompact** iff every sequence $\{x_n\}$ in $K$ has a convergent subsequence $\{x_{n_k}\}$, i.e.,

$$\rho(x_{n_k}, x) \to 0 \text{ as } n_k \to \infty \text{ for some } x \in X.$$  

**Definition (compact set)**

A subset $K$ of a metric space $X$ is called **compact** iff it is a closure of a precompact set.
Finite Dimensional Compactness

Theorem (Heine-Borel)

A set $K \subset \mathbb{R}^n$ is compact iff $K$ is both bounded and closed

Theorem (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

Examples:

1. $X = [0, 1]$ is compact (due to Heine-Borel Thm)
2. $X = \mathbb{R}$ is not compact because $M = \{1, 2, \ldots, n, \ldots\}$ does not have any convergent subsequence
3. $X = \ell_2$. A closed and bounded unit ball $B_1(0)$ is not compact. Indeed, consider the following sequence:

$$e_1 = \{1, 0, \ldots, 0, \ldots\}, e_2 = \{0, 1, 0, \ldots, 0, \ldots\}, e_3 = \{0, 0, 1, \ldots, 0, \ldots\}, \ldots$$

where $\rho(e_i, e_j) = \sqrt{2}$ if $i \neq j$. Hence, $\{e_i\}$ and any of its subsequence does not converge
Compactness (cont.)

Theorem (Cantor)

Consider a sequence \( K_1 \supset \ldots \supset K_2 \supset K_n \supset \ldots \) of nonempty compact subsets of a metric space \( X \). Then

\[
K = \bigcap_{i=1}^{\infty} K_i \neq \emptyset.
\]

Definition (\( \varepsilon \)-net of a set)

A subset \( N \) of a metric space \( X \) is called an \( \varepsilon \)-net of a set \( M \) iff

\[
\forall x \in M \ \exists y \in N \ \text{s.t.} \ \rho(x, y) < \varepsilon
\]

(In particular, \( M \) may coincide with \( X \))

Theorem

A continuous image \( f(K) \) of a compact set \( K \subset X \) is compact.
References