1-dim Sobolev Spaces, Parallelogram Law & Polarization Identity, Orthogonality

Week 07

October 6–10, 2014
Consider $u \in L^1[a, b]$

### Definition (weakly differentiable function)

A function $u \in L^1[a, b]$ is **weakly differentiable** iff there exists $v \in L^1[a, b]$ called a **weak derivative** of $u$ s.t.

$$
\int_0^1 u(t)\varphi'(t) \, dt = -\int_0^1 v(t)\varphi(t) \, dt, \quad \forall \varphi \in C_0^\infty[a, b]
$$

Denote that $v(t)$ by $u'(t)$

If function has a strong (classical) derivative then it is a weak derivative

This concept can be generalized further to $n$th weak derivative of $u$ by

$$
\exists \; v \in L^1[a, b] \text{ s.t. } \int_0^1 u(t)\varphi^{(n)}(t) \, dt = (-1)^n \int_0^1 v(t)\varphi(t) \, dt, \quad \forall \varphi \in C_0^\infty[a, b]
$$

Again, denote that $v(t)$ by $u^{(n)}(t)$
1-dim Sobolev Spaces (cont.)

Examples:

\[ f(x) = |x| \] is not differentiable on \([-1, 1]\) but has a weak derivative there which is

\[
g(x) = \begin{cases} 
-1, & x \in [-1, 0) \\
0, & x = 0 \\
1, & x \in (0, 1] 
\end{cases}
\]

A weak derivative is unique up to a set of measure zero.

All rules for classical derivatives (sum, product rules etc.) hold true in this case too.
1-dim Sobolev Spaces (cont.)

Define for $1 \leq p < \infty$:

$$W^{k,p}[a, b] = W^p_k[a, b] = \left\{ u \in L^p[a, b] : u^{(i)} \in L^p[a, b], \; i = 0, 1, \ldots, k \right\}$$

Note: $f(x) = |x| \in W^{1,2}[-1, 1]$

Define a norm

$$\| u \|_{k,p} = \| u \|_{W^{k,p}[a, b]} = \left( \sum_{i=0}^{k} \| u^{(i)} \|_{L^p[a, b]}^p \right)^{\frac{1}{p}}$$

and $W^{k,p}[a, b]$ is Banach with respect to this norm

Special spaces:

1. $W^{1,1}[a, b]$ – space of absolutely continuous on $[a, b]$ functions
2. $W^{1,\infty}[a, b]$ – space of Lipschitz continuous on $[a, b]$ functions
3. $W^{k,2} = H^k$ – Hilbert space
Corollaries of Cauchy-Schwarz inequality

**Corollary 1:**

Let \((X, (\cdot, \cdot))\) be an inner product space over \(\mathbb{K}\). Then \(\|x\| := (x, x)^{\frac{1}{2}}\) for \(x \in X\) is a norm on \(X\).

**Corollary 2:**

Let \((X, (\cdot, \cdot))\) be an inner product space, let \(\|x\| := (x, x)^{\frac{1}{2}}\). For given \(x, y \in X\) the equality \(\|x + y\| = \|x\| + \|y\|\) holds iff \(y = 0\) or \(x = \lambda y\) for some \(\lambda \geq 0\).

**Corollary 3:**

The inner product \((\cdot, \cdot)\) of an inner product space is a \(\mathbb{K}\)-valued continuous mapping on \(X \times X\), where the norm topology of \(X\) is determined by the inner product.

**Corollary 4:**

For \(x \in (X, (\cdot, \cdot))\):

\[
\|x\| = \sup_{\|y\| = 1} |(x, y)| = \sup_{\|y\| \leq 1} |(x, y)|
\]
Corollaries of Cauchy-Schwarz inequality (cont.)

Theorem (Parallelogram Law)

Let $X$ be an inner product space. Then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in X$$

Theorem (Polarization Identity)

Let $X$ be an inner product space over $\mathbb{K}$. Then for all $x, y \in X$,

$$(x, y) = \begin{cases} \frac{\|x + y\|^2}{2} - \frac{\|x - y\|^2}{2}, & \mathbb{K} = \mathbb{R} \\ \frac{\|x + y\|^2}{2} - \frac{\|x - y\|^2}{2} + i \left( \frac{\|x + iy\|^2}{2} - \frac{\|x - iy\|^2}{2} \right), & \mathbb{K} = \mathbb{C} \end{cases}$$
Theorem (inner product space characterization)

Let \((X, \| \cdot \|)\) be a normed space over \(K\). Then if any \(x, y \in X\), the parallelogram identity holds:

\[
\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2, \quad \forall x, y \in X,
\]

then \(X\) is an inner product space with an inner product defined by

\[
(x, y) = \begin{cases} \frac{x + y}{2} \cdot \frac{x + y}{2} - \frac{x - y}{2} \cdot \frac{x - y}{2}, & K = \mathbb{R} \\ \frac{x + y}{2} \cdot \frac{x + y}{2} - \frac{x - y}{2} \cdot \frac{x - y}{2} + i \left( \frac{x + iy}{2} \cdot \frac{x + iy}{2} - \frac{x - iy}{2} \cdot \frac{x - iy}{2} \right), & K = \mathbb{C} \end{cases}
\]

and \(\sqrt{(x, x)} = \| x \| \) for \( x \in X \)

Note: If \((X, (\cdot, \cdot))\) is an inner product space the inner product induces a norm on \(X\). We thus have the notions of convergence, completeness, compactness etc.
Orthogonality

Let $X$ be an inner product space, $x, y \in X$ and $M, N \subseteq X$ be subsets

**Definition (orthogonal vectors)**

$x$ and $y$ are called *orthogonal* iff $(x, y) = 0$ (notation: $x \perp y$)

**Definition (orthonormal vectors)**

$x$ and $y$ are called *orthonormal* iff $\|x\| = \|y\| = 1$ and $x \perp y$

**Definition (orthogonal sets)**

$M$ and $N$ are called *orthogonal* iff $(x, y) = 0$ for all $x \in M$, $y \in N$ (notation: $M \perp N$)

**Note:**

1. $M \perp N \Rightarrow M \cap N \subseteq \{0\}$
2. $x = 0$ is the only element orthogonal to every $y \in X \neq \emptyset$
Pythagorean Theorem

Definition (direct sum of sets)

Let $X$ be a linear space over $\mathbb{K}$, let $Y$ and $Z$ be linear subspaces of $X$. $X$ is the **direct sum** of $Y$ and $Z$ (notation: $Y \oplus Z$) iff

1. $\forall x \in X \exists y \in Y$ and $\exists z \in Z$ s.t. $x = y + z$
2. $Y \cap Z = \{0\}$

Theorem (Pythagorean)

Let $(X, (\cdot, \cdot))$ be an inner product space over $\mathbb{K}$. Let $x, y \in X$

1. If $\mathbb{K} = \mathbb{R}$ then $x \perp y$ iff $\|x + y\|^2 = \|x\|^2 + \|y\|^2$
2. If $\mathbb{K} = \mathbb{C}$ then
   (a) $(x, y) \in \mathbb{R}$ iff $\|x + iy\|^2 = \|x\|^2 + \|y\|^2$
   (b) $x \perp y$ iff $(x, y) \in \mathbb{R}$ and $\|x + y\|^2 = \|x\|^2 + \|y\|^2
References

- Hunter/Nachtergaele “Applied Analysis” pp. 93, 128–130, 133