Linear Bounded Operators

Week 11

November 3–7, 2014
Let $T : X \rightarrow Y$ be a linear map, where $X$, $Y$ are normed spaces. Then

1. If $\exists \ m > 0$ s.t. $m\|x\| \leq \|T(x)\|$, $x \in X$ then $T$ has a continuous inverse $T^{-1}$ defined on $\mathcal{R}(T)$, i.e. $T^{-1} : \mathcal{R}(T) \rightarrow X$ and
   \[
   \|T^{-1}(y)\| \leq \frac{1}{m}\|(y)\|, \quad \forall \ y \in \mathcal{R}(T)
   \]

2. If $\exists \ T^{-1} : \mathcal{R}(T) \rightarrow X$ then $\exists \ m > 0$ s.t. $m\|x\| \leq \|T(x)\|$, $\forall \ x \in X$

Note: we don’t require $T$ to be continuous. Also, Thm says nothing about $\mathcal{R}(T)$ (we don’t know if $T$ is onto or not)
Linear Operators (cont.)

Definition (linearly isomorphic spaces)
Two linear spaces $X$ and $Y$ are **linearly isomorphic** iff there exists a 1-to-1 onto linear map $T : X \to Y$.

Definition (topologically isomorphic spaces)
If $X$ and $Y$ are normed (linear) spaces and $T$ and $T^{-1}$ are linear bounded maps then $X$ and $Y$ are **topologically isomorphic**.

Definition (isometrically isomorphic spaces)
If $T$ also preserves norms, i.e. $\|T(x)\| = \|x\| \ \forall \ x \in X$, then $X$ and $Y$ are **isometrically isomorphic**.

Normally, “isomorphic” = “topologically isomorphic”.
Convergence of Operators

Definition (uniform convergence of operators)

Consider a sequence \( \{ T_n \}_{n=1}^{\infty} \) in \( B(X, Y) \). Then \( \{ T_n \}_{n=1}^{\infty} \) converges uniformly as \( n \to \infty \) iff 

\[
\lim_{n \to \infty} \| T_n - T \| = 0 \text{ for some } T \in B(X, Y)
\]

Definition (strong convergence of operators)

A sequence \( \{ T_n \}_{n=1}^{\infty} \) in \( B(X, Y) \) converges strongly as \( n \to \infty \) iff

\[
\forall x \in X : \lim_{n \to \infty} T_n x = T x
\]
Recall that the space of \textit{linear bounded} operators $\mathcal{B}(X, Y)$ is a linear space, where $X$ and $Y$ are linear also.

\begin{itemize}
  \item \textbf{Theorem}
  
  Let $X$ be a normed space and $Y$ a Banach space. A linear bounded operator $T$ defined on a linear dense set $L \subset X$ ($\overline{L} = X$) can be extended into the whole space $X$ without increasing its norm, i.e. \exists \ ! \hat{T} : X \to Y$ s.t. $\hat{T}x = Tx \ \forall \ x \in L$ and $\|\hat{T}\| = \|T\|$

  \item \textbf{Theorem}
  
  Let $X$ and $Y$ be normed spaces. If $Y$ is Banach then $\mathcal{B}(X, Y)$ is also \textit{Banach}

  \item \textbf{Theorem}
  
  Given a linear bounded operator $T : X \to X$, where $X$ is a complete space and $\|T\| \leq q < 1$. Then the operator $T + I$ has inverse $(T + I)^{-1}$, which is a \textit{linear bounded} operator
\end{itemize}
References

- Hunter/Nachtergaele “Applied Analysis” pp. 95–102, 104–105