Linear Bounded Operators

Week 12

November 10–14, 2014
**Space of Linear Bounded Operators**

**Theorem**

Given a linear bounded operator $T : X \to X$, where $X$ is a complete space and $\| T \| \leq q < 1$. Then the operator $T + I$ has inverse $(T + I)^{-1}$, which is a linear bounded operator.

**Definition (uniform convergence of operators)**

Consider a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{B}(X, Y)$. Then $\{T_n\}_{n=1}^{\infty}$ converges uniformly as $n \to \infty$ iff $\lim_{n \to \infty} \|T_n - T\| = 0$ for some $T \in \mathcal{B}(X, Y)$.

**Definition (strong convergence of operators)**

A sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{B}(X, Y)$ converges strongly as $n \to \infty$ iff $\forall x \in X : \lim_{n \to \infty} T_n x = T x$.

**Lemma**

If $T_n \to T$ uniformly then $T_n \to T$ strongly.

**Theorem**

Let $T_n \to T$ uniformly, where $T_n, T \in \mathcal{B}(X, Y)$ iff
Remark:
Often instead of strong convergence, one says about convergence pointwise.

Strong convergence $\not\Rightarrow$ uniform convergence

Consider space $\ell_2$ and mapping $y = P_n x$ where $\{y_n\}_{n=1}^{\infty}$ defined by

$$y_i = x_i, \quad i = 1, \ldots, n \quad \text{and} \quad y_i = 0 \quad \text{for} \quad i > n$$

Then

$$\|P_n x - x\|^2 = \sum_{i=n+1}^{\infty} |x_i|^2 \to 0 \Rightarrow P_n \to I \text{ strongly (i.e. } \forall x \in \ell_2).$$

However, for $x \in \ell_1$: $\|x\| = 1$ s.t. $P_n x = 0$ we have

$$\|x\| = \|P_n x - x\| = \|(P_n - I)x\| = 1$$

$$\Rightarrow \|P_n - I\| = \sup_{\|x\|=1} \geq 1,$$

i.e. there is no uniform convergence of $P_n$.
Space of Linear Bounded Operators (cont.)

**Lemma**

Let \( \{T_n\} \subset \mathcal{B}(X, Y) \), \( \exists \ C > 0 \) and \( \overline{B_r(x_0)} \) s.t. \( \| T_n x \| \leq C \) \( \forall x \in \overline{B_r(x_0)} \). Then \( \{\| T_n \|\} \) is bounded

**Theorem (Principle of Uniform Boundedness)**

If \( \{T_n x\} \) is bounded for all \( x \in X \) then \( \{\| T_n \|\} \) is bounded

**Theorem**

Let \( \{T_n\} \subset \mathcal{B}(X, Y) \) and \( T_n \to T \in \mathcal{B}(X, Y) \) strongly iff

1. \( \{\| T_n \|\} \) is bounded
2. \( T_n \to T \in \mathcal{B}(X, Y) \) strongly \( \forall x \in M, \overline{M} = X \)
References

Hunter/Nachtergaele “Applied Analysis” pp. 100–113