I. Approximation by Smooth Functions

Lecture 06

February 04, 2014
Sobolev Spaces (cont.)

We will consider the following properties of Sobolev spaces:

1. Approximation of Sobolev functions by smooth functions;
2. Embedding theorems;
3. Boundary values of Sobolev functions and Trace theorems;
4. Compactness results.
Local Approximation by Smooth Functions

Theorem

A function $f \in L^1_{loc}(\Omega)$ is weakly differentiable in $\Omega$ if and only if there is a sequence $\{f_n\}$ of functions $f_n \in C^\infty(\Omega)$ s.t. $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^1_{loc}(\Omega)$. Then the weak derivative of $f$ is given by $g = \partial^\alpha f \in L^1_{loc}(\Omega)$.

Theorem

Suppose that $f \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$ and set $f^\varepsilon = \eta^\varepsilon \ast f$ in $\Omega_{\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$. Then

1. $f^\varepsilon \in C^\infty(\Omega_{\varepsilon})$ for each $\varepsilon > 0$,
2. $f^\varepsilon \to f$ in $W^{k,p}_{loc}(\Omega)$ as $\varepsilon \to 0^+$. 

Theorem

Let $\Omega \subset \mathbb{R}^n$ be bounded, and suppose that $f \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exist functions $f_n \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ s.t. $f_n \to f$ in $W^{k,p}(\Omega)$ as $n \to \infty$.

In other words, $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

We do not assert that $f_n \in C^\infty(\bar{\Omega})$!

It is not true in general that if $\Omega \subset \mathbb{R}^n$ is a bounded domain then $C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. 
Global Approximation by Functions Smooth up to the Boundary

However if we assume additional regularity properties on $\Omega$, we obtain the following result:

**Theorem**

Let $\Omega \subset \mathbb{R}^n$ be bounded, and $\partial \Omega$ is $C^1$. Suppose $f \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exist functions $f_n \in C^\infty(\overline{\Omega})$ such that $f_n \rightarrow f$ in $W^{k,p}(\Omega)$ as $n \rightarrow \infty$. 
References

- Evans pp. 264–268