Euler-Lagrange Equations

Lecture 24

April 15, 2014
Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a **Lagrangian**, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$, and $1 < p < \infty$

Then for a given $\varphi \in W^{1,q}(\Omega)$, we study the variational functional

$$F(u) = \int_{\Omega} F(x, u, Du) \, dx \quad \text{for } u : \Omega \to \mathbb{R} \quad (1)$$

and look for *minimizers* of $F$ within a given class:

$$V = \left\{ v \in W^{1,q}(\Omega) : v - \varphi \in W^{1,q}_0(\Omega) \right\} \quad (2)$$

i.e. look for $u \in V$ s.t. $F(u) = \inf_{v \in V} F(v)$
Recall,

**Definition**

Associated with given by (1) functional $\mathcal{F}$ equations

$$ \frac{\partial F}{\partial u} - \frac{\partial}{\partial x_i} \frac{\partial F}{\partial p_i} = 0 \quad (3) $$

(where the Einstein summation convention is employed)

are called the **Euler-Lagrange equations**

**Definition**

We say that $u \in V$ with $V$ given by (2) is a **weak solution** of the associated Euler-Lagrange equations (3) if and only if

$$ \int_\Omega F_{p_i}(x, u, Du)D_i v + F_z(x, u, Du)v = 0 \quad \text{for all } v \in W_0^{1,q}(\Omega) $$
Next, we show that under suitable growth conditions on $F$, the minimizers of (1) do indeed solve the Euler-Lagrange equations.

**Lemma**

Suppose the Lagrangian $F$ satisfies the following growth condition: there exists $C > 0$ s.t. for all $x \in \Omega$, $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$:

1. $|F(x, z, p)| \leq C \left( |z|^{q-1} + |p|^{q-1} + 1 \right)$;
2. $|D_p F(x, z, p)| \leq C \left( |z|^{q-1} + |p|^{q-1} + 1 \right)$;
3. $|D_z F(x, z, p)| \leq C \left( |z|^{q-1} + |p|^{q-1} + 1 \right)$.

If $u \in V$ is a minimizer of $\mathcal{F}(\cdot)$ then $u$ is a weak solution to the associated Euler-Lagrange equations (3).
Remark about Solutions of EL Equations (3)

Remark:
In general, EL eq. (3) may have other solutions that do not correspond to minima of $\mathcal{F}(\cdot)$. However, in special case that joint mapping $(z, p) \mapsto L(x, z, p)$ is convex for each $x \in \Omega$, then every weak solution of (3) is in fact a minimizer of $\mathcal{F}(\cdot)$

To demonstrate this, let $u \in V$ solve (3) in weak sense and select any $v \in V$. Using convexity of $(z, p) \mapsto L(x, z, p)$ we obtain

$$F(x, v, p) \geq F(x, z, p) + D_z F(x, z, p) \cdot (v - z) + D_p F(x, z, p) \cdot (q - p)$$

Set $z = u(x)$, $v = v(x)$, $p = Du(x)$, $q = Dv(x)$, integrate over $\Omega$:

$$\mathcal{F}(v) \geq \mathcal{F}(u) + \int_\Omega [D_z F(x, u, Du) \cdot (v - u) + D_p F(x, u, Du) \cdot (Dv - Du)] \, dx$$

From (3) the second term of LHS is zero, hence, $\mathcal{F}(u) \leq \mathcal{F}(v)$ for all $v \in V$
Regularity of Minimizers

First, of all, select $H^1(\Omega)$ as the underling functional space. Then make the following simplifying assumptions on the Lagrangian $F$:

1. $F$ is a function only of $p$: $F = F(p)$;
2. $F$ is at least $C^2$;
3. $|D_p^2 F(p)| \leq C$ for all $p \in \mathbb{R}^n$ (this is stronger than the condition required in Lemma above), i.e. $|D_p F(p)| \leq C(|p| + 1)$;
4. $F$ is uniformly convex, i.e. there exists $\lambda > 0$ s.t. for all $\xi \in \mathbb{R}^n$: $F_{p_ip_j}(p)\xi_i\xi_j \geq \lambda|\xi|^2$

Let, as above, for a given $\varphi \in H^1(\Omega)$, the space $V$ is defined by

$$V = \{ v \in H^1(\Omega) : v - \varphi \in H^1_0(\Omega) \}$$

Then by Lemma above, a minimizer $u \in V$ is a weak sol’n of EL:

$$D_i (F_{p_i}(Du)) = 0 \quad \text{in } \Omega,$$

i.e. for each $v \in H^1_0(\Omega)$ one has $\int_{\Omega} F_{p_i}(Du)D_i v = 0$
Theorem (Interior $H^2(\Omega)$ Regularity)

Under assumptions above, suppose $u \in H^1(\Omega)$ is a weak solution to the Euler-Lagrange equation (5) in $\Omega$. Then $u \in H^2_{loc}(\Omega)$. 
References

- Evans pp. 472–475