Homogenization of Stokes Equation. Darcy Law

Week 09

October 19 – 23, 2015
Setting

We are going to derive the Darcy’s Law for an incompressible viscous fluid flowing in porous medium.

Start with the steady Stokes equation in a periodic porous medium, with no-slip (Dirichlet) boundary condition on the solid pores.

Then the Darcy’s Law is rigorously obtained by periodic homogenization using the two-scale asymptotic method.

Denote by $\varepsilon > 0$ the ratio of the period to the overall size of the porous medium. The pores are usually much smaller than the characteristic length if the reservoir, hence, $\varepsilon \ll 1$.

The porous medium is contained in a domain $\Omega$ and the fluid part is $\Omega_\varepsilon$, i.e. $\Omega_\varepsilon$ is periodically perforated domain and has many small holes of size $O(\varepsilon)$, which represent solid obstacles that the fluid cannot penetrate.

Denote by

- $u_\varepsilon \rightsquigarrow$ velocity of fluid
- $p_\varepsilon \rightsquigarrow$ pressure of fluid
- $f \rightsquigarrow$ density of forces acting on the fluid
The motion of the fluid in $\Omega_\varepsilon$ is governed by the steady Stokes, complemented with Dirichlet BC:

**microscopic Stokes equations**

\[
\begin{aligned}
\nabla p_\varepsilon - \mu \varepsilon^2 \Delta u_\varepsilon &= f, \quad x \in \Omega_\varepsilon \\
\nabla \cdot u_\varepsilon &= 0, \quad x \in \Omega_\varepsilon \\
\n\quad u_\varepsilon &= 0, \quad x \in \partial \Omega_\varepsilon
\end{aligned}
\]

(1)

where the fluid viscosity is a positive number $\mu$ that is scaled by factor $\varepsilon^2$ so that the velocity $u_\varepsilon$ has a non-trivial limit as $\varepsilon \to 0$

Physically, the very small viscosity $O(\varepsilon^2)$ balances the friction of the fluid on the solid boundaries due to no-slip BC

For existence and uniqueness, we require $f \in L^2(\Omega)^d \Rightarrow \exists ! u_\varepsilon \in H^1_0(\Omega_\varepsilon)^d$, and $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$ solving (1)

**Definition**

The *homogenization* of (1) is to find the effective equation satisfied by the limits of as $u_\varepsilon, p_\varepsilon$ as $\varepsilon \to 0$
Setting (cont.)

Describe here assumptions on porous domain $\Omega_\varepsilon$:

Periodic structure is determined by $\Omega$ and an associated microstructure or periodic cell $Y = [0, 1]^d$, which is made of two parts

$$
Y_f \rightsquigarrow \text{fluid} \\
Y_s \rightsquigarrow \text{solid}
$$

s.t. $Y_f \cup Y_s = Y$, and $Y_f \cap Y_s = \emptyset$

Assume $\Omega$ is smooth, bounded, connected set in $\mathbb{R}^d$ and $Y_f$ is a smooth and connected open subset of $Y$

Domain $\Omega$ is covered with a regular mesh of size $0 < \varepsilon \ll 1$

Each cell $Y_\varepsilon^e$ of type $(0, \varepsilon)^d$ and is divided into

$$
Y_{f,i}^\varepsilon \rightsquigarrow \text{fluid part} \\
Y_{s,i}^\varepsilon \rightsquigarrow \text{solid part}
$$

The the fluid domain $\Omega_\varepsilon$ is

$$
\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} Y_{s,i}^\varepsilon \cap \left[ \bigcup_{i=1}^{N(\varepsilon)} Y_{f,i}^\varepsilon \right]
$$
Finally, note that the sequence of solutions \((u_\varepsilon, p_\varepsilon)\) is not defined in a fixed domain independent of \(\varepsilon\) but rather in a varying set \(\Omega_\varepsilon\). To state the HOMOGENIZATION THM, convergences in a fixed Sobolev spaces (defined on \(\Omega\)) are used, which requires that \((u_\varepsilon, p_\varepsilon)\) be extended to the whole domain \(\Omega\). An extension \((\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)\) of \((u_\varepsilon, p_\varepsilon)\) is defined in \(\Omega\) and coincides with \((u_\varepsilon, p_\varepsilon)\) on \(\Omega_\varepsilon\).
Homogenization Theory

**Theorem**

There exists an extension \((\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)\) of solution \((u_\varepsilon, p_\varepsilon)\) of (1) s.t.

\[
\tilde{u}_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(\Omega)^d
\]

\[
\tilde{p}_\varepsilon \to p \quad \text{in} \quad L^2(\Omega)/\mathbb{R}
\]

where \((u, p)\) is the unique solution of the homogenized problem called the **Darcy’s Law**:

\[
\begin{aligned}
    u(x) & = \frac{1}{\mu} A(f(x) - \nabla p(x)) , & x \in \Omega \\
    \nabla \cdot u(x) & = 0 , & x \in \Omega \\
    u(x) \cdot n & = 0 , & x \in \partial \Omega
\end{aligned}
\]

where \(A\) is a symmetric, positive-definite tensor, called the **effective permeability** defined by

\[
A_{ij} = \int_{Y_f} \nabla \lambda_i(y) \cdot \nabla \lambda_j(y) \, dy
\]

where \(\lambda_i(y)\) is the unique solution in \(H^1_\#(Y_f)^d\) of the **cell problem**

\[
\begin{aligned}
    \nabla_y q_i(y) - \Delta_y \lambda_i(y) & = e_i , & y \in Y_f \\
    \nabla_y \cdot \lambda_i(y) & = 0 , & y \in Y_f \\
    \lambda_i(y) & = 0 , & y \in Y_s
\end{aligned}
\]

with \(\{e_1, \ldots, e_d\}\) being a canonical basis of \(\mathbb{R}^d\)
The weak convergence of the velocity can be further improved by the following corrector result.

As the same notation as in THM, the velocity satisfies

$$\tilde{u}_\varepsilon(x) - \sum_{i=1}^{d} \lambda_i \left( \frac{x}{\varepsilon} \right) u_i(x) \to 0 \quad \text{in } L^2(\Omega)^d$$

where $u_i$ are the components of $u$ in the THM, and $\{\lambda_1, \ldots, \lambda_d\}$ are solutions of the cell problems (4).

Remark:

The homogenized problem (2) is the Darcy’s Law, that is, the flow rate $u$ is proportional to the balance of forces including pressure. The permeability tensor $A$ depends only on microstructure, $Y_f$ of the porous medium but not on exterior forces, nor on physical properties of the fluid.